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The complex Toda chains and the simple Lie algebras: II. Explicit solutions and asymptotic behaviour

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Abstract. We propose a compact and explicit expression for the solutions of the complex Toda chains related to the classical series of simple Lie algebras $\mathfrak{g}$. The solutions are parametrized by a minimal set of scattering data for the corresponding Lax matrix. They are expressed as sums over the weight systems of the fundamental representations of $\mathfrak{g}$ and are explicitly covariant under the corresponding Weyl group action. In deriving these results we start from the Moser formula for the $\boldsymbol{A}_r$ series and obtain the results for the other classical series of Lie algebras by imposing appropriate involutions on the scattering data. Thus we also show how Moser's solution goes into that of Olshanetsky and Perelomov. The results for the large-time asymptotics of the $\boldsymbol{A}_r$-CTC solutions are extended to the other classical series $\boldsymbol{B}_r$–$\boldsymbol{D}_r$. We exhibit also some 'irregular' solutions for the $\boldsymbol{D}_{2n+1}$ algebras whose asymptotic regimes at $t \to \pm\infty$ are qualitatively different. Interesting examples of bounded and periodic solutions are presented and the relations between the solutions for the algebras $\boldsymbol{D}_4$, $\boldsymbol{B}_3$ and $\boldsymbol{G}_2$ are analysed.

1. Introduction

The famous Toda chain model [1–12] was initially introduced in order to study nearest-neighbour interactions in atomic chains. Soon it was shown that it also possesses interesting mathematical properties and that to each simple Lie algebra $\mathfrak{g}$ one can relate a natural generalization to the Toda chain [5–7, 9–11, 13–16], namely

$$\frac{\mathrm{d}^2\vec{q}}{\mathrm{d}t^2} = \sum_{k=1}^{r} \alpha_k \mathrm{e}^{-(\vec{q},\alpha_k)} \tag{1.1}$$

where $\vec{q} = (q_1, \ldots, q_r)$ is a vector in the root space $\mathbb{E}^r$ of the algebra $\mathfrak{g}$ of rank r and α_k, $k = 1, 2, \ldots, r$ are the simple roots of $\mathfrak{g}$. One may view $q_k(t)$ as the coordinate of the kth particle and study the effect of their interaction. A number of results in this direction are known showing how the (real) Toda chain (RTC) (1.1) can be viewed as a completely integrable Hamiltonian system and how it can be solved explicitly, see [2–7, 9–11, 13, 14, 16].

Recently, it became known that generalizing the RTC model with $\mathfrak{g} \simeq sl(N)$ to 'complex' particles (i.e. now $q_k(t)$ become complex-valued functions) allows one to describe the interactions in the N-soliton trains of the nonlinear Schrödinger equation in the adiabatic approximation. In this case each soliton behaves as a separate entity ('particle'); $\operatorname{Re} q_k(t)$ describes its centre-of-mass position and $\operatorname{Im} q_k(t)$ determines its phase, for more details see [17–20]. These facts draw our interest towards the study of the complex Toda chain (CTC) models when the dynamical variables $q_k(t)$ become complex, while the time variable t stays real.

It is well known that a large number of results for the RTC are trivially generalized to the CTC case by just making the corresponding parameters complex. These include the Lax pairs and the explicit solutions. However, since each 'complex' particle has two degrees of freedom their interaction becomes much more complex and qualitatively different compared to the real case. In particular, the set of asymptotic regimes for the CTC is much richer than those of RTC. In addition to the asymptotically free particle regime (the only one possible for RTC), CTC also allows for bound state regimes, mixed regimes, degenerate regimes, etc. These facts were reported in [17–20, 34] and are compatible with the well known ones, see [21–23, 27–33, 35, 36].

The present paper is a natural extension of [20]; it also contains proofs and generalizations of the results in [20].

There are several methods for solving the RTC which also readily generalize for CTC. The method in [6, 13, 14, 16] allows one to write down the solution as

$$(\vec{q}(t), \omega_k) - (\vec{q}(0), \omega_k) = \ln\langle\omega_k|\mathrm{e}^{-2L(0)t}|\omega_k\rangle \tag{1.2}$$

where ω_k is the kth fundamental weight of $\mathfrak{g}$, $|\omega_k\rangle$ is the highest weight vector in the kth fundamental representation $R(\omega_k)$ and $L(0)$ is the Lax matrix evaluated at $t = 0$, see formula (2.1a) below. The right-hand side of (1.2) has the obvious advantage of being written in compact and invariant form. However, it is difficult to extract from it explicitly parametrized solutions.

Another well known approach to solving the Toda chain models was developed in [11]. It allows one to express the solution in terms of $2r$ constants. Starting from a comparatively simple expression for $X_1 = \exp(-q_1(t))$ one then calculates $X_k = \exp(-q_k(t))$ as a determinant of a $k \times k$ matrix whose elements are determined by the derivatives of X_1, see [7, 11]. One may also use a recurrent procedure to evaluate X_k. However, this leads to rather complicated and difficult to analyse expressions.

Our first aim in the present paper will be to analyse (1.2) and write it down in the form

$$(\vec{q}(t), \omega_k) = \ln \mathcal{B}_{\mathfrak{g};k}(t) \tag{1.3a}$$

$$\mathcal{B}_{\mathfrak{g};k}(t) = \sum_{\gamma\in\Gamma_{\mathfrak{g}}(\omega_k)} \exp\left[-2(\gamma, \vec{\zeta})t + (\vec{\varphi}_0, \gamma)\right] W^{(k)}(\vec{\zeta}, \gamma). \tag{1.3b}$$

Here $(\gamma, \vec{\zeta})$ is the scalar product between the vector $\vec{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_r)$ and the weight $\gamma \in \Gamma_{\mathfrak{g}}(\omega_k)$; ζ_s are eigenvalues of $L(0)$ and we suppose that they satisfy $\zeta_k \neq \zeta_j$ for $k \neq j$. The components of the vectors $\vec{\zeta}$ and $\vec{\varphi}_0 = (\varphi_{01}, \varphi_{02}, \ldots, \varphi_{0r})$ provide the $2r$ (complex) parameters directly related to the minimal set of scattering data $\mathcal{T}_{\mathfrak{g}}$ of $L(0)$, see formula (2.15) below; $\Gamma_{\mathfrak{g}}(\omega_k)$ is the set of weights of the kth fundamental representation of $\mathfrak{g}$; $W^{(k)}(\vec{\zeta}, \gamma)$ are t-independent functions which are defined in section 3 below. Thus the right-hand side of (1.3) like the right-hand side of (1.2) is invariant and at the same time is explicitly parametrized. This fact allows us to calculate explicitly the large-time asymptotics of $\vec{q}(t)$:

$$\lim_{t\to\pm\infty} (\vec{q}(t) - \vec{v}^{\pm}t) = \vec{\varphi}_0^{\pm} + \vec{\beta}^{\pm}. \tag{1.4}$$

It is a well known fact [15, 16] that $\vec{v}^{+} = w_0(\vec{v}^{-}) = -2\vec{\zeta}$, where w_0 is the Weyl group element which maps the highest weight ω_k of the kth fundamental representation $R(\omega_k)$ of $\mathfrak{g}$ into the corresponding lowest-weight vector ω_k^{-}. We provide explicit expressions for $\vec{\beta}^{\pm}$ as functions of $\vec{\zeta}$ and also show that

$$\vec{\beta}^{+}(\vec{\zeta}) = \vec{\beta}^{-}(w_0(\vec{\zeta})) \qquad \vec{\varphi}_0^{+} = w_0(\vec{\varphi}_0^{-}) = \vec{\varphi}_0. \tag{1.5}$$

In fact the solutions to the CTC related to a certain simple Lie algebra $\mathfrak{g}$ may be derived in two ways. The first one is to cast the solution (1.2) in the form

$$\langle\omega_k|\mathrm{e}^{-2L(0)t}|\omega_k\rangle = \sum_{\gamma\in\Gamma_{\mathfrak{g}}(\omega_k)} (\langle\omega_k|V|\gamma\rangle)^2 \mathrm{e}^{-2(\vec{\zeta},\gamma)t} \tag{1.6}$$

and then try to evaluate the matrix elements $\langle\omega_k|V|\gamma\rangle$ in terms of the scattering data $\mathcal{T}_{\mathfrak{g}}$. This requires the explicit construction of V for each of the fundamental representations $R(\omega_k)$ of $\mathfrak{g}$.

The second possibility which will be used below, is to start with the well known solution of Moser [4] for $sl(N)$ with conveniently chosen N and impose on the scattering data the involution that restricts it to $\mathfrak{g}$. Obviously both solutions must coincide. The proof of this fact is also one of the results in the present paper.

In the next section we introduce the notation and analyse the properties of the fundamental representations of the classical series of simple Lie algebras and derive some useful relations between the matrix elements of the typical and the other fundamental representations. In section 3 we prove formula (1.3) for each of the classical series $\boldsymbol{A}_r$–$\boldsymbol{D}_r$. In section 4 we also extend the results for the large-time asymptotics of the $\boldsymbol{A}_r$-CTC solutions to the other classical series $\boldsymbol{B}_r$–$\boldsymbol{D}_r$. We also exhibit some 'irregular' solutions for the $\boldsymbol{D}_{2n+1}$ algebras whose asymptotic regimes at $t \to \pm\infty$ are qualitatively different. We also provide some interesting examples of bounded and periodic solutions and analyse the relations between the solutions for the algebras $\boldsymbol{D}_4$, $\boldsymbol{B}_3$ and $\boldsymbol{G}_2$.

2. Preliminaries

In what follows we shall use the so-called 'symmetric' Lax representation for the CTC model (1.1):

$$L(t) = \sum_{k=1}^{r} \left(b_k H_k + a_k(E_{\alpha_k} + E_{-\alpha_k})\right) \tag{2.1a}$$

$$M(t) = \sum_{k=1}^{r} a_k(E_{\alpha_k} - E_{-\alpha_k}) \tag{2.1b}$$

where $a_k = \frac{1}{2}\mathrm{e}^{-(q,\alpha_k)/2}$ and $b_k = \frac{1}{2}\mathrm{d}q_k/\mathrm{d}t$. For $\mathfrak{g} \simeq sl(N)$ we have $a_k = \frac{1}{2}\mathrm{e}^{(q_{k+1}-q_k)/2}$. It is well known that to each root α from the root system $\Delta_g \subset \mathbb{E}^r$ one can relate the element H_α of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Analogously, to $q(t) = \mathrm{Re}\,q(t) + \mathrm{i}\,\mathrm{Im}\,q(t) \in \mathfrak{h}$ there corresponds the vector $\vec{q}(t) = \mathrm{Re}\,\vec{q}(t) + \mathrm{i}\,\mathrm{Im}\,\vec{q}(t)$, whose real and imaginary parts are vectors in the root space $\mathbb{E}^r$.

The integrals of motion in involution for the CTC model are provided by the eigenvalues, $\zeta_k = \kappa_k + \mathrm{i}\eta_k$, of L. The solutions of both the CTC and the RTC are determined by the scattering data for $L(0)$. When the spectrum of $L(0)$ is non-degenerate, i.e. $\zeta_k \neq \zeta_j$ for $k \neq j$, then this scattering data consists of

$$\mathcal{T} \equiv \{\zeta_1, \ldots, \zeta_N, r_1, \ldots, r_N\} \tag{2.2}$$

where r_k are the first components of the corresponding eigenvectors $v^{(k)}$ of $L(0)$ in the typical representation $R(\omega_1)$ of $\mathfrak{g}$, $N = \dim R(\omega_1)$. If we combine all eigenvectors $v^{(k)}$ as columns of the matrix V then $r_k = V_{1k}$ and

$$L(0)V = VZ \qquad Z = \mathrm{diag}\,(\zeta_1, \ldots, \zeta_N). \tag{2.3}$$

It is known that the eigenvectors of symmetric matrices $L(0)$ with a non-degenerate spectrum can always be normalized, i.e. following [4, 11, 12] we require that

$$(v^{(k)}, v^{(k)}) \equiv \sum_{s=1}^{N} (V_{sk})^2 = 1 \qquad k = 1, \ldots, N \tag{2.4}$$

and besides $V^T = V^{-1}$. Equation (2.4) determines r_k up to a sign.

From (2.3) it follows that

$$G(t) \equiv \mathrm{e}^{-2L(0)t} = V \mathrm{e}^{-2Zt} V^{-1} \tag{2.5}$$

and we can rewrite (1.2) in the form (1.3) with

$$\mathcal{B}_{\mathfrak{g};k}(t) = \sum_{\gamma \in \Gamma_{\mathfrak{g}}(\omega_k)} (\langle \omega_k | V | \gamma \rangle)^2 \mathrm{e}^{-2(\vec{\zeta}, \gamma)t}. \tag{2.6}$$

Thus our aim will be realized if we obtain explicit expressions for the matrix elements $\langle \omega_k | V | \gamma \rangle$ of V in the kth fundamental representation in terms of $\mathcal{T}$ which is determined by the spectral data of $L(0)$ in the typical representation $R(\omega_1)$ of $\mathfrak{g}$.

The eigenvalues of $L(0)$, and especially their real parts κ_k, which can be calculated directly from the initial conditions as will become evident below, uniquely determine the asymptotic behaviour of the solutions [20]. We will use this fact extensively for the description of the different types of asymptotic behaviour.

The minimal set of scattering data for $\mathfrak{g} \sim sl(N)$ is obtained from (2.2) by imposing on $\mathcal{T}$ the restrictions $\sum_{k=1}^{N} \zeta_k = 0$ and

$$\sum_{k=1}^{N} r_k^2 = 1 \tag{2.7}$$

which follows from $V^T = V^{-1}$. Therefore, one may consider as $\mathcal{T}_{\boldsymbol{A}_r}$ the set

$$\begin{aligned} &\mathcal{T}_{A_r} \equiv \{\zeta_1, \ldots, \zeta_N; \tilde{\varphi}_{01}, \ldots, \tilde{\varphi}_{0N}\} \qquad N = r + 1 \\ &\tilde{\varphi}_{0k} = \ln r_k^2 - \frac{1}{N} \sum_{s=1}^{N} \ln r_s^2. \end{aligned} \tag{2.8}$$

Although the number of elements in $\mathcal{T}_{\boldsymbol{A}_r}$ is $2N$ (instead of $2r = 2N - 2$) it is obvious that only $2r$ of them are independent.

For the other classical series of simple Lie algebras the elements of $\mathcal{T}$ (2.2) satisfy symmetry relations, namely [20]

$$\zeta_k = -\zeta_{\bar{k}} \qquad \bar{k} = N + 1 - k \tag{2.9}$$

$$r_k r_{\bar{k}} = \mathrm{e}^{-q_1(0)} w_k \tag{2.10}$$

for $k = 1, \ldots, N$ where N is the dimension of the typical representation $R(\omega_1)$ and the value of $q_1(t)$ at $t = 0$ is determined through the normalization condition (2.7). The coefficients w_k are time independent and are expressed in terms of $\zeta_1, \ldots, \zeta_r$ as follows, see appendix A.

$\boldsymbol{B}_r$ *series:* $N = 2r + 1$. Note that in this case $\zeta_{r+1} = 0$,

$$w_k = \frac{1}{8\zeta_k^2} \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2} \tag{2.11}$$

and in addition to (2.10),

$$r_{r+1}^2 = \mathrm{e}^{-q_1(0)} w_{r+1} \qquad w_{r+1} = \prod_{s=1}^{r} \frac{1}{4\zeta_s^2}. \tag{2.12}$$

Inserting (2.10)–(2.12) into (2.7) we obtain a quadratic equation for $\exp(-q_1(0))$, so it can be expressed in terms of $\mathcal{T}_g$.

$\boldsymbol{C}_r$ *series:* $N = 2r$ and

$$w_k = -\frac{1}{4\zeta_k} \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2}. \tag{2.13}$$

$\boldsymbol{D}_r$ *series:* $N = 2r$ and

$$w_k = \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2}. \tag{2.14}$$

In the last two cases again $\exp(-q_1(0))$ is determined from (2.7). The derivation of the solution for the $\boldsymbol{D}_r$ series requires some additional efforts. The main problem here is related to the treatment of the spinor representations.

The proof of (2.11)–(2.14) is based on the study of the properties of the corresponding matrices V and is given in the appendix A. Then one easily finds that the set of parameters

$$\mathcal{T}_{\mathfrak{g}} \equiv \{\zeta_1, \ldots, \zeta_r; \varphi_{01}, \ldots, \varphi_{0r}\} \qquad \varphi_{0k} = \ln(r_k/r_{\bar{k}}) \tag{2.15}$$

uniquely determines $\mathcal{T}$ (2.2), which in turn provides the full set of eigenvalues and eigenvectors of $L(0)$.

Next we will need a number of details from the representation theory of the simple Lie algebras. In what follows by $R_{\mathfrak{g}}(\omega)$ we will denote the representation of $\mathfrak{g}$ with highest weight ω; $\Gamma_{\mathfrak{g}}(\omega)$ stands for the set of weights of $R_{\mathfrak{g}}(\omega)$. Often when the choice for $\mathfrak{g}$ is clear from the context, we will omit the subscript and will write simply $\Gamma(\omega)$ and $R(\omega)$. We will also need to introduce ordering not only in the root system $\Delta_{\mathfrak{g}}$ but also in the weight system $R_{\mathfrak{g}}(\omega)$. To this end we will use a vector $\vec{K}$ in the root space $\mathbb{E}^r$ such that $(\gamma_1 - \gamma_2, \vec{K}) \neq 0$ and $(\alpha - \beta, \vec{K}) \neq 0$ for any two weights $\gamma_1 \neq \gamma_2 \in \Gamma_{\mathfrak{g}}(\omega)$ and roots $\alpha \neq \beta \in \Delta_{\mathfrak{g}}$. Without restrictions we can choose $\vec{K}$, together with the vector $-\vec{\kappa} = -\operatorname{Re}\vec{\zeta}$ to be in the fundamental Weyl chamber so that $(\omega - \gamma, \vec{K}) > 0$ for any $\gamma \in \Gamma_{\mathfrak{g}}(\omega)$.

Let us now denote by γ_k, $k = 1, \ldots, N$ the set of weights of the typical representation $\Gamma_{\mathfrak{g}}(\omega_1)$ of $\mathfrak{g}$, namely

$$\gamma_k = e_k - \frac{1}{r+1} \sum_{a=1}^{r+1} e_a \qquad N = r+1 \tag{2.16a}$$

for $\mathfrak{g} \simeq \boldsymbol{A}_r$,

$$\gamma_k = \begin{cases} e_k & \text{for} \quad 1 \leqslant k \leqslant r \\ -e_{\bar{k}} & \text{for} \quad r+1 \leqslant k \leqslant 2r \end{cases} \qquad N = 2r \tag{2.16b}$$

for $\mathfrak{g} \simeq \boldsymbol{C}_r, \boldsymbol{D}_r$,

$$\gamma_k = \begin{cases} e_k & \text{for} \quad 1 \leqslant k \leqslant r \\ 0 & \text{for} \quad k = r+1 \\ -e_{\bar{k}} & \text{for} \quad r+2 \leqslant k \leqslant 2r+1 \end{cases} \qquad N = 2r+1 \tag{2.16c}$$

for $\mathfrak{g} \simeq \boldsymbol{B}_r$; in all formulae above $\bar{k} = N + 1 - k$. The corresponding weight vectors specify an orthonormal basis in $R_{\mathfrak{g}}(\omega_1)$ and will be denoted by $|\gamma_k\rangle$.

An important and well known tool to construct the fundamental representations of $\mathfrak{g}$ is to make use of the exterior tensor products of $R_{\mathfrak{g}}(\omega_1)$. Indeed, the orthonormal basis in $\wedge^k R_{\mathfrak{g}}(\omega_1)$ consists of the weight vectors

$$|\gamma_I\rangle \equiv |\gamma_{i_1,i_2,\dots,i_k}\rangle = |\gamma_{i_1} \wedge \gamma_{i_2} \wedge \dots \wedge \gamma_{i_k}\rangle \qquad I \equiv \{i_1, i_2, \dots, i_k\} \tag{2.17}$$

with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$; the weight corresponding to (2.17) is obviously

$$\gamma_I = \gamma_{i_1} + \gamma_{i_2} + \cdots + \gamma_{i_k}. \tag{2.18}$$

Here and in what follows we shall use the one-to-one correspondence between the set of indices I and the corresponding weight γ_I and weight vector $|\gamma_I\rangle$.

For $\mathfrak{g} \simeq \boldsymbol{A}_r$ all fundamental representations are, in fact, exterior powers of the typical one $R(\omega_1)$:

$$R(\omega_k) = \wedge^k R(\omega_1) \tag{2.19}$$

for $k = 1, 2, \dots, r = N - 1$ and $\gamma_I = \gamma_I^{(k)}$ where the upper index k means that $\gamma_I^{(k)} \in \Gamma(\omega_k)$. We remind also other well known fact, namely that in $\wedge^k R(\omega_1)$ we have

$$\langle\omega_k|V|\gamma_I^{(k)}\rangle \equiv \langle\gamma_1 \wedge \gamma_2 \wedge \dots \wedge \gamma_k|V|\gamma_{i_1} \wedge \gamma_{i_2} \wedge \dots \wedge \gamma_{i_k}\rangle = V \left\{ \begin{array}{cccc} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{array} \right\} \tag{2.20}$$

where

$$V \left\{ \begin{array}{cccc} j_1 & j_2 & \dots & j_k \\ i_1 & i_2 & \dots & i_k \end{array} \right\}$$

is the minor of the group element $V \in \mathfrak{G}$ determined by the intersection of the rows $j_1, j_2, \dots, j_k$ with the columns $i_1, i_2, \dots, i_k$. Thus given a group element $V \in SL(r+1)$ in the typical representation $R(\omega_1)$ one can construct its image for each of the fundamental representations $R(\omega_k)$, $k = 1, 2, \dots, r$.

Let us now explain how this can be done for the other simple Lie algebras of the classical series. To this end we shall make use of the well known facts about the root systems [24, 25] of $\mathfrak{g}$ and about the tensor products of their fundamental representations, see [26].

Let us now consider the $\boldsymbol{B}_r$ series. Then we have

$$\begin{aligned} \wedge^k R(\omega_1) &= R(\omega_k) \\ \wedge^r R(\omega_1) &= R(2\omega_r) \end{aligned} \qquad \text{for} \quad k = 1, 2, \dots, r-1 \tag{2.21}$$

where

$$\omega_k = e_1 + e_2 + \cdots + e_k \qquad \omega_r = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_r). \tag{2.22}$$

Here ω_r is the highest weight of the spinor representation of $\boldsymbol{B}_r$. Therefore, the relations (2.20) also hold for $\mathfrak{g} \simeq \boldsymbol{B}_r$ with $k = 1, 2, \dots, r-1$.

Another well known fact is that the symmetric tensor product of $R(\omega_r)$ is generically reducible and

$$R(\omega_r) \underset{S}{\otimes} R(\omega_r) = R(2\omega_r) \oplus \sum_{i=1}^{[r/4]} R(\omega_{r-4i+1}) \oplus R(\omega_{r-4i}) \tag{2.23}$$

where $\omega_0 = 0$ and $R(\omega_0)$ is the trivial one-dimensional representation of $\mathfrak{g}$.

We now have two possibilities to introduce a basis in $R(2\omega_r)$. The first one is to use (2.17) and (2.21) like above. The second possibility is to argue that $R(\omega_r) \underset{S}{\otimes} R(\omega_r)$ is spanned by

$$|\gamma_I^{(r)} \otimes \gamma_I^{(r)}\rangle \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\big(|\gamma_I^{(r)} \otimes \gamma_J^{(r)}\rangle + |\gamma_J^{(r)} \otimes \gamma_I^{(r)}\rangle\big) \tag{2.24}$$

where $\gamma_I^{(r)} \in \Gamma(\omega_r)$ are weights of the spinor representation of $\boldsymbol{B}_r$. Obviously they have the form

$$\gamma_I^{(r)} = \tfrac{1}{2}\sum_{k=1}^{r} \sigma_k e_k = \tfrac{1}{2}\sum_{k=1}^{r} \gamma_{i_k} \tag{2.25}$$

where $\sigma_k = \pm 1$. The corresponding sets I (and J) now must be special in the sense that: (a) $r+1 \notin I$; (b) if $i_k \in I$ then $\bar{i}_k = N+1-i_k \notin I$. In other words, I does not contain pairs of 'conjugated' indices. To the weight vector $|\gamma_I^{(r)} \otimes \gamma_I^{(r)}\rangle$ there corresponds the weight $2\gamma_I^{(r)}$ which can be obtained from $2\omega_r$ by a Weyl group transformation. This means that $|\gamma_I^{(r)} \otimes \gamma_I^{(r)}\rangle$ belongs to the representation $R(2\omega_r)$ in the right-hand side of (2.23). Thus we find

$$\begin{aligned} V\left\{\begin{array}{cccc} 1 & 2 & \dots & r \\ i_1 & i_2 & \dots & i_r \end{array}\right\} &= \langle 2\omega_r|V|2\gamma_I^{(r)}\rangle \\ &= \langle \omega_r \otimes \omega_r|V|\gamma_I^{(r)} \otimes \gamma_I^{(r)}\rangle = \big(\langle \omega_r|V|\gamma_I^{(r)}\rangle\big)^2. \end{aligned} \tag{2.26}$$

Next we choose $\mathfrak{g} \simeq \boldsymbol{C}_r$ series. It is well known that the exterior products of $R(\omega_1)$ generically are reducible, namely

$$\begin{aligned} &\wedge^k R(\omega_1) = R(\omega_k) \oplus \wedge^{k-2} R(\omega_1) \\ &\omega_k = e_1 + e_2 + \cdots + e_k \qquad \text{for} \quad k = 2, \dots, r \end{aligned} \tag{2.27}$$

but the relations (2.17), (2.18) and (2.20) also hold for $\mathfrak{g} \simeq \boldsymbol{C}_r$ with $k = 2, \dots, r$. Equation (2.27) reflects the existence of a non-trivial invariant subspace in $\wedge^2 R(\omega_1)$ (see [24])

$$|c\rangle = \sum_{k=1}^{N} (-1)^{k+1} |\gamma_k \wedge \gamma_{\bar{k}}\rangle = \sum_{k,m=1}^{N} S_{km} |\gamma_k \wedge \gamma_m\rangle \tag{2.28}$$

where S is the matrix entering into the definition of the symplectic group (see (A.5) below); namely

$$X \in Sp(2r) \quad \leftrightarrow \quad S X^T S^{-1} = X^{-1}. \tag{2.29}$$

It is easy to check that due to (2.29) we have $X|c\rangle = |c\rangle$ for any element $X \in Sp(2r)$. Thus we establish that c determines the one-dimensional invariant subspace in $\wedge^2 R(\omega_1) = (\mathbb{C}|c\rangle) \oplus R(\omega_2)$.

Let us now analyse the weights in the weight systems $\Gamma(\omega_k)$ and their multiplicities. The highest weight $e_1 + \cdots + e_k$ has multiplicity 1; the corresponding set of indices is $I = \{1, 2, \dots, k\}$. Let us consider next the weight $\gamma_{(1)} = e_1 + \cdots + e_{k-2}$; to it there corresponds each of the following sets of indices $I = \{1, 2, \dots, k-2, p, \bar{p}\}$, $k-1 \leqslant p \leqslant r$. Therefore, to $\gamma_{(1)}$ there corresponds the subspace $V(\gamma_{(1)}) \subset \wedge^k R(\omega_1)$ which is spanned by the vectors $|\gamma_1 \wedge \dots \wedge \gamma_{k-2} \wedge \gamma_p \wedge \gamma_{\bar{p}}\rangle$ and has dimension $\dim V(\gamma_{(1)}) = r-k+2$. At the same time the Freudenthal formula shows that the multiplicity of $\gamma_{(1)} = e_1 + \cdots + e_{k-2}$ in $R(\omega_k)$ is $r-k+1$. This difference is due to the fact that in $V(\gamma_{(1)})$ there exist a one-dimensional invariant subspace determined by $|\gamma_1 \wedge \dots \wedge \gamma_{k-2} \wedge c\rangle$. The same argument can be applied to each of the weights $\gamma'_{(1)} = w\gamma_{(1)}$ where w is an element of the Weyl group. Indeed, it is known that the Weyl group preserves: (a) the lengths of the weights, i.e. $(\gamma_{(1)}, \gamma_{(1)}) = (\gamma'_{(1)}, \gamma'_{(1)}) = k-2$; (b) the multiplicities of the weights. In fact, the Weyl group is isomorphic to the group of permutations $\mathcal{S}_{2r}$ of the indices $\{1, 2, \ldots, \bar{2}, \bar{1}\}$; therefore, instead of looking at the transformed weight $\gamma'_{(1)}$ we may consider the corresponding set of indices which can be obtained from I by applying a specific element of $\mathcal{S}_{2r}$. This analysis can also be continued by considering weights of length $k-4$, e.g. $\gamma_{(2)} = e_1 + \cdots e_{k-4}$. Skipping the details we formulate the result, namely

$$\wedge^k R(\omega_1) = R(\omega_k) \oplus (\mathbb{C}|c\rangle) \wedge (\wedge^{k-2} R(\omega_1)). \tag{2.30}$$

It remains only to note that from (2.20), (2.30) and from $V|c\rangle = |c\rangle$ there follows

$$\begin{aligned} \langle \omega_k | V | \gamma_{i_1} \wedge \ldots \wedge \gamma_{i_k} \rangle \equiv V \begin{Bmatrix} 1 & \ldots & k \\ i_1 & \ldots & i_k \end{Bmatrix} & \\ = \langle \omega_k | V | \gamma_I^{(k)} \rangle + \rho \langle \omega_k | V | c \wedge \gamma_{I'} \rangle & \\ = \langle \omega_k | V | \gamma_I^{(k)} \rangle & \end{aligned} \tag{2.31}$$

where ρ is some constant, $\gamma_I^{(k)} \in R(\omega_k)$ and $\gamma_{I'} \in \wedge^{k-2} R(\omega_1)$. In the last line we also used the fact that the representations $R(\omega_k)$ and $\wedge^{k-2} R(\omega_1)$ span mutually orthogonal subspaces of $\wedge^k R(\omega_1)$.

Finally, let $\mathfrak{g} \simeq \boldsymbol{D}_r$ series. Then we have

$$\wedge^k R(\omega_1) = R(\omega_k) \qquad \text{for} \quad k = 1, 2, \ldots, r-2 \tag{2.32a}$$

$$\wedge^{r-1} R(\omega_1) = R(\omega_{r-1} + \omega_r) \tag{2.32b}$$

$$\wedge^r R(\omega_1) = R(2\omega_r) \oplus R(2\omega_{r-1}) \tag{2.32c}$$

where

$$\omega_k = e_1 + e_2 + \cdots + e_k \qquad \text{for} \quad k = 1, 2, \ldots, r-2 \tag{2.33a}$$

$$\omega_{r-1} = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_{r-1} - e_r) \tag{2.33b}$$

$$\omega_r = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_{r-1} + e_r). \tag{2.33c}$$

From (2.32*a*) and (2.33*a*) we find that the relations (2.20) also hold for $\mathfrak{g} \simeq \boldsymbol{D}_r$ with $k = 1, 2, \ldots, r-2$. Let us now analyse the spinor representations $R(\omega_{r-1})$ and $R(\omega_r)$ of $\boldsymbol{D}_r$. It is known [26] that

$$R(\omega_r) \otimes R(\omega_{r-1}) = R(\omega_{r-1} + \omega_r) \oplus \sum_{i=1}^{[(r-1)/2]} R(\omega_{r-2i-1}) \tag{2.34a}$$

$$R(\omega_r) \underset{S}{\otimes} R(\omega_r) = R(2\omega_r) \oplus \sum_{i=1}^{[r/4]} R(\omega_{r-4i}). \tag{2.34b}$$

where $\underset{S}{\otimes}$ denotes the symmetrized tensor product and $R(\omega_0)$ is the trivial one-dimensional representation of $\mathfrak{g}$.

The basis in $R(\omega_r) \otimes R(\omega_{r-1})$ is determined by $|\gamma_I^{(r)} \otimes \gamma_{I'}^{(r-1)}\rangle$ where $\gamma_I^{(r)} \in \Gamma(\omega_r)$ and $\gamma_{I'}^{(r-1)} \in \Gamma(\omega_{r-1})$. The sets of indices $I = \{i_1, \ldots, i_r\}$ and $I' = \{i'_1, \ldots, i'_r\}$ are related to the weights $\gamma_I^{(r)}$ and $\gamma_{I'}^{(r-1)}$ by

$$\gamma_{I'}^{(r-1)} = \tfrac{1}{2} \sum_{k=1}^{r} \sigma'_k e_k = \tfrac{1}{2} \sum_{k=1}^{r} \gamma_{i'_k} \qquad \gamma_I^{(r)} = \tfrac{1}{2} \sum_{k=1}^{r} \sigma_k e_k = \tfrac{1}{2} \sum_{k=1}^{r} \gamma_{i_k} \tag{2.35}$$

where σ_k and σ_k' take values ± 1 and

$$\prod_{k=1}^{r} \sigma_k' = -1 \qquad \text{and} \qquad \prod_{k=1}^{r} \sigma_k = 1. \tag{2.36}$$

Let us now consider a pair of weights $\gamma_I^{(r)}$ and $\gamma_{I'}^{(r-1)}$ such that $\sigma_k = \sigma_k'$ for all but one value of k, i.e. $I \cap I' = \{j_1, \dots, j_{r-1}\}$. Then the vectors $\gamma_I^{(r)} + \gamma_{I'}^{(r-1)}$ and $\omega_r + \omega_{r-1}$ have the same length and one can check that they are related by a Weyl group transformation. Therefore, $|\gamma_I^{(r)} \otimes \gamma_{I'}^{(r-1)}\rangle$ belongs to the representation $R(\omega_{r-1} + \omega_r)$ in (2.34*a*) and

$$\begin{aligned} V \left\{ \begin{array}{cccc} 1 & 2 & \dots & r-1 \\ i_1 & i_2 & \dots & i_{r-1} \end{array} \right\} &= \langle \omega_{r-1} + \omega_r | V | \gamma_{I'}^{(r-1)} + \gamma_I^{(r)} \rangle \\ &= \langle \omega_{r-1} | V | \gamma_{I'}^{(r-1)} \rangle \langle \omega_r | V | \gamma_I^{(r)} \rangle. \end{aligned} \tag{2.37a}$$

Finally, the spinor representation $R(\omega_r)$ is considered quite analogously to that of the $\boldsymbol{B}_r$ case. There is only one additional fact that should be taken care of: now the weights $2\omega_r$ and $2\omega_{r-1}$ have the same length but are related by an outer-automorphism rather than by a Weyl group element. In order to separate the weights belonging to the modules $R(2\omega_r)$ and $R(2\omega_{r-1})$ we need projectors onto each of these invariant subspaces in $\wedge^r R(\omega_r)$ in (2.32*c*). As we shall see below, we will need to sort out only the weights of length r which are given by either $2\gamma_I^{(r-1)}$ or by $2\gamma_I^{(r)}$. The projectors which will separate these two types of weights can be constructed by using (2.36) and the fact that $\zeta_{\bar{k}} = -\zeta_k$; the matrix elements of these projectors are given by

$$f_I^{\pm} \equiv f_{i_1, i_2, \dots, i_r}^{\pm} = \frac{1}{2}\left(1 \pm \frac{\zeta_1 \zeta_2 \dots \zeta_r}{\zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_r}} \right). \tag{2.37b}$$

Indeed, it is easy to check now that

$$f_I^{+} \gamma_{I'}^{(r-1)} = f_{I'}^{-} \gamma_I^{(r)} = 0 \qquad f_I^{+} \gamma_I^{(r)} = \gamma_I^{(r)} \qquad f_{I'}^{-} \gamma_{I'}^{(r-1)} = \gamma_{I'}^{(r-1)}. \tag{2.37c}$$

Thus we obtain the relation

$$f_{i_1, i_2, \dots, i_r}^{+} V \left\{ \begin{array}{cccc} 1 & 2 & \dots & r \\ i_1 & i_2 & \dots & i_r \end{array} \right\} = \langle 2\omega_r | V | 2\gamma_I^{(r)} \rangle = (\langle \omega_r | V | \gamma_I^{(r)} \rangle)^2. \tag{2.37d}$$

From (2.37*a*)–(2.37*d*) we find the necessary expressions for $\langle \omega_{r-1} | V | \gamma_I^{(r-1)} \rangle$ and $\langle \omega_r | V | \gamma_I^{(r)} \rangle$ through the minors of V, see (A.14).

3. The solutions of the CTC revisited

We will now analyse the structure of the solutions to the CTC for each of the classical series of Lie algebras. Our aim will be to write them down in the form (1.3) and calculate the functions $W^{(k)}(\vec{\zeta}, \gamma)$ for each of the classical series separately.

3.1. The $\boldsymbol{A}_r$ series

It is well known [4, 6, 12, 33] that the solutions in this case can be expressed through the principal minors of the group element $G(t)$ (2.5). Using the Binet–Cauchy formula, or equivalently (2.20) and (A.3) we obtain the functions

$$A_1(t) = \sum_{k=1}^{N} r_k^2 \mathrm{e}^{-2\zeta_k t} \tag{3.1}$$

$$A_k(t) = \sum_{i_1<i_2<\cdots<i_k} r_{i_1}^2 r_{i_2}^2 \ldots r_{i_k}^2 \mathrm{e}^{-2(\zeta_{i_1}+\zeta_{i_2}\ldots+\zeta_{i_k})t} W^2(i_1, i_2, \ldots, i_k) \tag{3.2}$$

$$A_N(t) = \prod_{s=1}^{N} r_s^2 W^2(1, 2, \ldots, N) = \mathrm{e}^{-Nq_1(0)} \tag{3.3}$$

which are proportional to the above-mentioned principal minors of $G(t)$. Here ζ_k, r_k are the scattering data of the Lax matrix $L(0)$ introduced in (2.2), (2.7) and $W_I(\vec{\zeta}) \equiv W(i_1, i_2, \ldots, i_k)$ is the Vandermonde determinant:

$$W_I(\vec{\zeta}) \equiv W(i_1, i_2, \ldots, i_k) = \prod_{s>p;\, s,p\in I} 2(\zeta_s - \zeta_p). \tag{3.4}$$

Note that each of the factors in the right-hand side of (3.4) can be viewed as the scalar product $(\vec{\zeta}, \alpha)$ where α is an appropriately chosen root $\alpha \in \Delta_{\mathfrak{g}}$. This and the fact that $(\vec{\zeta}, w_0(\alpha)) = (w_0(\vec{\zeta}), \alpha)$ make it easy to introduce in a natural way the action of the Weyl group element w_0 on $W_I(\vec{\zeta})$, namely $w_0 : W_I(\vec{\zeta}) \to W_I(w_0(\vec{\zeta}))$.

The solution of the corresponding CTC is then given by

$$(\vec{q}(t) - \vec{q}(0), \omega_k) = \ln A_k(t) \tag{3.5}$$

or

$$q_k(t) = q_1(0) + \ln \frac{A_k(t)}{A_{k-1}(t)} \tag{3.6}$$

with $A_0 = 1$ and $q_1(0)$ defined by (3.3).

For further convenience we introduce the functions

$$B_k(t) = \mathrm{e}^{kq_1(0)} A_k(t) \tag{3.7}$$

and rewrite the solution to the CTC in the form

$$(\vec{q}(t), \omega_k) = \ln B_k(t). \tag{3.8}$$

Now introducing the set of variables (2.8) and taking into account (2.16a) we easily cast the solution (3.8) into the form (1.3) with

$$B_k(t) = \mathcal{B}_k(t) = \sum_{\gamma_I^{(k)}\in\Gamma(\omega_k)} \exp\left(\vec{\tilde{\varphi}}(t), \gamma_I^{(k)}\right) W_I^2(\vec{\zeta})(W(1, 2, \ldots, N))^{-2k/N} \tag{3.9}$$

$$\vec{\tilde{\varphi}}(t) = (\tilde{\varphi}_1(t), \ldots, \tilde{\varphi}_N(t)) \qquad \tilde{\varphi}_k(t) = -2\zeta_k t + \tilde{\varphi}_{0k} \tag{3.10}$$

where $\tilde{\varphi}_{0k}$ were introduced in (2.8). Another possibility is to use $\vec{\varphi}(t) = -2\vec{\zeta}t + \vec{\varphi}_0$ with

$$\varphi_{0k} = \tilde{\varphi}_{0k} - \ln w_k - \frac{2}{N} \ln W(1, \ldots, N) = q_1(0) + \ln \frac{r_k^2}{w_k}$$

$$w_k = \prod_{s=1}^{k-1} \frac{1}{2(\zeta_k - \zeta_s)} \prod_{s=k+1}^{N} \frac{1}{2(\zeta_s - \zeta_k)} \tag{3.11}$$

which gives

$$\mathcal{B}_{\boldsymbol{A}_r;k}(t) = \sum_{i_1<i_2<\cdots<i_r}^{N} \rho_{i_1}^2 \rho_{i_2}^2 \ldots \rho_{i_k}^2 \mathrm{e}^{-2(\zeta_{i_1}+\cdots+\zeta_{i_k})t} W_I^2(\vec{\zeta}) w_{i_1} \ldots w_{i_k} \tag{3.12}$$

with $\rho_k = \mathrm{e}^{\varphi_{0k}}$. Then finally $\mathcal{B}_{\boldsymbol{A}_r;k}(t)$ can be rewritten as

$$\mathcal{B}_{\boldsymbol{A}_r;k}(t) = \sum_{\gamma_I^{(k)}\in\Gamma(\omega_k)} \exp\bigl(\vec{\varphi}(t), \gamma_I^{(k)}\bigr) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) \tag{3.13}$$

with

$$W^{(1)}(\zeta, \gamma_k) = w_k \qquad W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) = W_I^2(\vec{\zeta}) \prod_{s\in I} w_s \quad k = 2, 3, \ldots, r. \tag{3.14}$$

Now it is more natural to consider

$$\mathcal{T}_{\boldsymbol{A}_r} \equiv \{\zeta_1, \ldots, \zeta_N, \varphi_{01}, \ldots, \varphi_{0N}\} \qquad N = r + 1 \tag{3.15}$$

rather than (2.8), as the minimal set of scattering data for the $\boldsymbol{A}_r$ algebra case. Indeed, the elements of (3.15), like the ones of (2.8), naturally satisfy the identities

$$\sum_{k=1}^{N} \zeta_k = (\vec{\zeta}, \vec{\epsilon}) = 0 \qquad \sum_{k=1}^{N} \varphi_{0k} = (\vec{\varphi}, \vec{\epsilon}) = 0 \tag{3.16}$$

$\epsilon = e_1 + \cdots + e_N$, which is directly related to the fact that all the roots of $\boldsymbol{A}_r$ also satisfy $(\alpha, \vec{\epsilon}) = 0$.

However, equation (3.15) also has the advantage that imposing on its elements the symmetry condition (2.9) and

$$\varphi_{0k} = -\varphi_{0\bar{k}} \qquad \bar{k} = N + 1 - k \tag{3.17}$$

one can obtain the minimal set of scattering data for the $\boldsymbol{B}_r$ and $\boldsymbol{C}_r$ series.

Indeed, let us choose $\mathfrak{g} \simeq \boldsymbol{A}_{2r}$ and let us impose on $\mathcal{T}_{\boldsymbol{A}_{2r}}$ (2.9) and (3.17). First, note that from (2.9) and (3.11) with $N = 2r + 1$ we immediately find that w_k takes the form (2.11) for $k = 1, \ldots, r$ and (2.12) for $k = r + 1$; besides one can check that $w_{\bar{k}} = w_k$.

Next, the condition (3.17) with $\varphi_{0,\bar{k}} = q_1(0) + \ln(r_{\bar{k}}^2/w_k)$ leads immediately to the relation (2.10). Expressing from it $q_1(0)$ as $\ln(w_k/(r_k r_{\bar{k}}))$ we easily obtain $\varphi_{0,k} = \ln(r_k/r_{\bar{k}})$ for $k = 1, \ldots, r$ and $\varphi_{0,r+1} = 0$. Thus we showed that $\mathcal{T}_{\boldsymbol{A}_{2r}}$ with (2.9) and (3.17) reduces to (2.15) for the series $\boldsymbol{B}_r$.

The same procedure applied to $\mathcal{T}_{\boldsymbol{A}_{2r-1}}$ provides (2.15) for the series $\boldsymbol{C}_r$.

3.2. The $\boldsymbol{B}_r$ series

The solution is obtained from the one for the $\boldsymbol{A}_{2r}$ series by imposing the relations (2.9)–(2.12). Due to this only the first r of the functions $A_k(t)$ in (3.5) are independent. From the analysis in section 2 we see that only the expression for $A_r(t)$ requires additional special treatment†, see (2.26). Thus we find

$$A_k(t) = \mathrm{e}^{-kq_1(0)} \mathcal{B}_k(t) \qquad \text{for} \quad k = 1, \ldots, r-1 \tag{3.18}$$

$$A_r(t) = \mathrm{e}^{-rq_1(0)} \mathcal{B}_r^2(t). \tag{3.19}$$

† This is related to the existence of the spinor representation.

where $\mathcal{B}_k(t)$ are of the form (1.3*b*),

$$\mathcal{B}_k(t) = \sum_{\gamma_I^{(k)}\in\Gamma(\omega_k)} \exp\bigl(\vec{\varphi}(t), \gamma_I^{(k)}\bigr) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) \tag{3.20}$$

with

$$W^{(1)}(\zeta, \gamma_k) = w_k \qquad W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) = W_I^2(\vec{\zeta}) \prod_{s\in I} w_s \qquad \vec{\varphi}(t) = -2\vec{\zeta}t + \vec{\varphi}_0 \tag{3.21}$$

for $k = 1, \ldots, r-1$ and

$$W^{(r)}(\vec{\zeta}, \gamma_I^{(r)}) = W_I(\vec{\zeta}) \prod_{s=1}^{r} w_s^{1/2}. \tag{3.22}$$

Here $W_I(\vec{\zeta})$ is the Vandermonde determinant (3.4). We remind the reader that to each weight $\gamma_I^{(k)} \in \Gamma(\omega_k)$ there corresponds an ordered set of indices $I = \{i_1 < \cdots < i_k\}$ and that w_k are defined by (2.11) and (2.12). Each set I uniquely defines $\gamma_I^{(k)}$. If the weight $\gamma_I^{(k)}$ has a multiplicity greater than 1 then there exist several sets of indices related to $\gamma_I^{(k)}$. For example, the weights $e_1+\cdots+e_k$ and $e_1+\cdots+e_{k-1}$ in $\Gamma(\omega_k)$ have multiplicities 1; the corresponding sets of indices are $\{1, 2, \ldots, k\}$ and $\{1, 2, \ldots, k-1, r+1\}$, respectively. The weight $e_1+\cdots+e_{k-2}$, however, has multiplicity $r-k+2$; one may assign to it each of the following sets of indices $\{1, 2, \ldots, k-2, p, \bar{p}\}$ where $k-1 \leqslant p \leqslant r$. Analogously to the weight $e_1+\cdots+e_{k-3}$ and $e_1+\cdots+e_{k-4}$ with multiplicities $r-k+3$ and $\binom{r-k+4}{4}$ one can assign each of the sets $\{1, \ldots, k-3, p, r+1, \bar{p}\}$ with $k-2 \leqslant p \leqslant r$ and $\{1, \ldots, k-4, p_1, p_2, \bar{p}_1, \bar{p}_2\}$ with $k-3 \leqslant p_1 < p_2 \leqslant r$. In other words, we find that the number of sets I corresponds to the number of weights provided each weight is counted as many times as its multiplicity.

We note that $\mathcal{B}_k(t)$ can also be written in the form

$$\mathcal{B}_k(t) = \sum_{\gamma_I^{(k)}>0} 2\cosh\bigl(\vec{\varphi}(t), \gamma_I^{(k)}\bigr) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) + \sum_{I:\gamma_I^{(k)}=0} W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) \tag{3.23}$$

where the second sum runs over the sets of indices corresponding to the weight equal to zero. For the $\boldsymbol{B}_r$ series these sets are of the form

$$\{p_1, \ldots, p_s, \bar{p}_1, \ldots, \bar{p}_s\} \qquad \text{for} \quad k = 2s$$

and

$$\{p_1, \ldots, p_s, r+1, \bar{p}_1, \ldots, \bar{p}_s\} \qquad \text{for} \quad k = 2s+1.$$

Formula (3.23) reflects the fact that $\Gamma(\omega_k)$ is symmetric in the sense that if $\gamma_I^{(k)} \in \Gamma(\omega_k)$ then $-\gamma_I^{(k)} = w_0(\gamma_I^{(k)}) \in \Gamma(\omega_k)$. One can also check that

$$W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) = W^{(k)}(w_0(\vec{\zeta}), w_0(\gamma_I^{(k)})) = W^{(k)}(\vec{\zeta}, \gamma_{\bar{I}}^{(k)}) \qquad \bar{I} = \{\bar{i}_k, \ldots, \bar{i}_1\}.$$

3.3. The $\boldsymbol{C}_r$ series

Remark 1. When one imposes the symmetry condition (2.9) and (2.10) on the $\boldsymbol{A}_{2r-1}$ CTC the corresponding system of equations is slightly different from (1.1). The difference consists only in the coefficient of the term $\mathrm{e}^{-(\vec{q},\alpha_r)}$ which comes out as $\alpha_r/2$ instead of α_r. The extra $\frac{1}{2}$ factor is easy to take into account and therefore for the $\boldsymbol{C}_r$ series the relation between $q_k(t)$ and $\mathcal{B}_k(t)$ is slightly different, namely

$$q_k(t) = \ln \frac{\mathcal{B}_k(t)}{\mathcal{B}_{k-1}(t)} + \frac{1}{2}\ln 2. \tag{3.24}$$

Now we insert (2.13) into (3.1) and (3.2) to obtain the solutions for the $\boldsymbol{C}_r$ series. Thus we obtain

$$A_k(t) = \mathrm{e}^{-kq_1(0)} B_k(t) \tag{3.25}$$

for all $k = 1, 2, \ldots, r$ where

$$B_k(t) = \sum_{i_1<\cdots<i_k} \exp\left(\vec{\varphi}(t), \gamma_{i_1\ldots i_k}\right) W^{(k)}(\vec{\zeta}, \gamma_{i_1\ldots i_k}) \qquad \vec{\varphi}(t) = -2\vec{\zeta}t + \vec{\varphi}_0 \tag{3.26}$$

$$W^{(1)}(\zeta, \gamma_k) = w_k \qquad W^{(k)}(\vec{\zeta}, \gamma_{i_1\ldots i_k}) = W^2(i_1, \ldots, i_k) \prod_{s=1}^{k} w_{i_s} \tag{3.27}$$

and $\gamma_{i_1\ldots i_k}$ is a weight in $\wedge^k R(\omega_1)$. From our analysis in section 2 (see (2.30)) each of the weights $\gamma_{i_1,\ldots,i_k} = \gamma_{i_1} + \cdots + \gamma_{i_k}$ can be split into $|\gamma_{i_1,\ldots,i_k}\rangle = |\gamma_I^{(k)}\rangle + \rho|c \wedge \gamma_{I'}\rangle$ (see (2.31)) and we have

$$\langle\omega_k|V|\gamma_{i_1,\ldots,i_k}\rangle \equiv V\left\{\begin{matrix} 1 & \ldots & k \\ i_1 & \ldots & i_k \end{matrix}\right\} = \langle\omega_k|V|\gamma_I^{(k)}\rangle. \tag{3.28}$$

This means that the summation over all sets of indices $i_1 < i_2 < \cdots < i_k$ in (3.26) reduces to the sum over the weights of $\Gamma(\omega_k)$ only,

$$\mathcal{B}_k(t) = B_k(t) = \sum_{\gamma_I^{(k)}\in\Gamma(\omega_k)} \exp\left(\vec{\varphi}(t), \gamma_I^{(k)}\right) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) \tag{3.29}$$

$$W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) = W_I^2(\vec{\zeta}) \prod_{s\in I} w_s \qquad k = 2, 3, \ldots, r \tag{3.30}$$

i.e. we cast the solution in the form (1.3). Besides, as for the $\boldsymbol{B}_r$ series again $w_0(\gamma_I^{(k)}) = -\gamma_I^{(k)}$ so we have

$$\mathcal{B}_k(t) = \sum_{\gamma_I^{(k)}>0} 2\cosh\left(\vec{\varphi}(t), \gamma_I^{(k)}\right) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) + \sum_{I:\gamma_I^{(k)}=0} W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}). \tag{3.31}$$

We note that the sets of indices corresponding to $\gamma_I^{(k)} = 0$ are given by $\{p_1, \ldots, p_s, \bar{p}_1, \ldots, \bar{p}_s\}$ for $k = 2s$ and by the empty set for $k = 2s + 1$.

3.4. The $\boldsymbol{D}_r$ series

There are some differences in treating this case due to the fact that the Lax matrix $L(0)$ (A.8) is not a tridiagonal one. Nevertheless, the Moser formula (3.1)–(3.3) for $N = 2r$, together with the corresponding involution (2.9), (2.10), (2.14) provides the solution to $\boldsymbol{D}_r$. Due to the somewhat different structure of the eigenmatrix V we find that the first $r - 1$ functions $A_k(t)$ are given by (3.1), (3.2) and only $A_r(t)$ must be replaced by [20]

$$\tilde{A}_r(t) = \sum_{i_1<\cdots<i_r} r_{i_1}^2 \ldots r_{i_r}^2 \exp\left(-2(\zeta_{i_1} + \cdots + \zeta_{i_r})t\right) (f^+_{i_1,\ldots,i_r})^2 W^2(i_1, \ldots, i_r). \tag{3.32}$$

Besides, the $\boldsymbol{D}_r$ algebras have two spinor representations which require additional care. Note also that due to (A.14) the projector $f^+_{i_1\ldots i_r}$ (2.37b) enters in a natural way into $\tilde{A}_r$. Thus in the right-hand side only the terms related to the weights of $R(2\omega_r)$ give non-vanishing contributions.

Let us introduce the variables (2.15) and along with (3.7) for $k = 1, \ldots, r-1$ let us put $B_r(t) = \mathrm{e}^{rq_1(0)}\tilde{A}_r(t)$; then we can rewrite the solution for the $\boldsymbol{D}_r$ series in the form

$$(\vec{q}(t), \omega_k) = \ln B_k(t) \qquad \text{for} \quad k = 1, \ldots, r-2 \tag{3.33}$$

$$(\vec{q}(t), \omega_{r-1} + \omega_r) = \ln B_{r-1}(t) \qquad (\vec{q}(t), 2\omega_r) = \ln B_r(t). \tag{3.34}$$

Our analysis in section 2 shows that $w_k = w_{\bar{k}}$ and

$$\begin{aligned} B_k(t) &= \mathcal{B}_k(t) \qquad &&\text{for} \quad k = 1, \ldots, r-2 \\ B_{r-1}(t) &= \mathcal{B}_{r-1}(t)\mathcal{B}_r(t) \qquad && B_r(t) = \mathcal{B}_r(t)^2 \end{aligned} \tag{3.35}$$

with

$$\mathcal{B}_k(t) = \sum_{\gamma_I^{(k)} \in \Gamma(\omega_k)} \exp\big(\vec{\varphi}(t), \gamma_I^{(k)}\big) W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) \qquad \text{for} \quad k = 1, \ldots, r. \tag{3.36}$$

Here

$$W^{(1)}(\zeta, \gamma_k) = w_k \qquad \vec{\varphi}(t) = -2\vec{\zeta}t + \vec{\varphi}_0 \tag{3.37}$$

$$W^{(k)}(\vec{\zeta}, \gamma_I^{(k)}) = W_I^2(\vec{\zeta}) \prod_{s \in I} w_s \qquad k = 1, \ldots, r-2 \tag{3.38}$$

$$W^{(r-a)}(\vec{\zeta}, \gamma_I^{(r-a)}) = \sqrt{w_1 \ldots w_r}\, W_I(\vec{\zeta}) \qquad a = 0, 1 \tag{3.39}$$

and the set of indices $I = \{i_1, \ldots, i_r\}$ in (3.39) is such that it determines uniquely the weight $\gamma_I^{(r-a)} \in \Gamma(\omega_{r-a})$ in the corresponding spinor representation. In addition,

$$(\vec{q}(t), \omega_{r-a}) = \ln \mathcal{B}_{r-a}(t). \tag{3.40}$$

The explicit solutions for RTC with the simplest choices for $\mathfrak{g}$ with rank 2 in invariant form were proposed in the monograph [11]; they coincide with the particular cases of the ones given above provided a proper identification of the variables is performed. Here we choose to provide as examples the solution to the $\boldsymbol{D}_4$ case and its relation to the solutions for $\boldsymbol{B}_3$ and $\boldsymbol{G}_2$.

Example 1. *Let $\mathfrak{g} \simeq \boldsymbol{D}_4 = so(8)$. Then it has three eight-dimensional representations: $R(\omega_1)$ and the two spinor ones $R(\omega_3)$ and $R(\omega_4)$. The representation $R(\omega_2)$ is of dimension 28. We also remind the reader that $\boldsymbol{D}_4$ has an outer-automorphism v_1 of order three which interchanges the eight-dimensional representations; more precisely,*

$$\begin{aligned} v_1 &: \alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1 \qquad && v_1\alpha_2 = \alpha_2 \\ v_1 &: \omega_1 \to \omega_3 \to \omega_4 \to \omega_1 \qquad && v_1\omega_2 = \omega_2. \end{aligned} \tag{3.41}$$

The equations of motion for the $\boldsymbol{D}_4$-CTC have the form

$$\begin{aligned} q_{1,tt} &= \mathrm{e}^{q_2 - q_1} \qquad && q_{2,tt} = \mathrm{e}^{q_3 - q_2} - \mathrm{e}^{q_2 - q_1} \\ q_{3,tt} &= -\mathrm{e}^{q_3 - q_2} + \mathrm{e}^{q_4 - q_3} + \mathrm{e}^{-q_3 - q_4} \qquad && q_{4,tt} = -\mathrm{e}^{q_4 - q_3} + \mathrm{e}^{-q_3 - q_4}. \end{aligned} \tag{3.42}$$

The solution is provided by

$$\begin{aligned} q_1(t) &= \ln \mathcal{B}_1(t) \qquad && q_2(t) = \ln \frac{\mathcal{B}_2(t)}{\mathcal{B}_1(t)} \\ q_3(t) &= \ln \frac{\mathcal{B}_3(t)\mathcal{B}_4(t)}{\mathcal{B}_2(t)} \qquad && q_4(t) = \ln \frac{\mathcal{B}_4(t)}{\mathcal{B}_3(t)} \end{aligned} \tag{3.43}$$

where

$$\mathcal{B}_1(t) = 2\sum_{k=1}^{4} w_k \cosh\varphi_k(t) \tag{3.44a}$$

$$\mathcal{B}_2(t) = 8\sum_{i<j}^{4} w_i w_j \left[(\zeta_j - \zeta_i)^2 \cosh(\varphi_i + \varphi_j) + (\zeta_j + \zeta_i)^2 \cosh(\varphi_i - \varphi_j)\right] + 16\sum_{i=1}^{4} w_i^2\zeta_i^2 \tag{3.44b}$$

$$\begin{aligned}\mathcal{B}_3(t) = 2^{-11}\prod_{i<j}^{4}\frac{1}{\zeta_i^2-\zeta_j^2}\Big\{ & W(1,2,3,\bar{4})\cosh\tfrac{1}{2}(\varphi_1+\varphi_2+\varphi_3-\varphi_4)\\ &+W(1,2,4,\bar{3})\cosh\tfrac{1}{2}(\varphi_1+\varphi_2-\varphi_3+\varphi_4)\\ &+W(1,3,4,\bar{2})\cosh\tfrac{1}{2}(\varphi_1-\varphi_2+\varphi_3+\varphi_4)\\ &+W(2,3,4,\bar{1})\cosh\tfrac{1}{2}(\varphi_1-\varphi_2-\varphi_3-\varphi_4)\Big\}\end{aligned} \tag{3.44c}$$

$$\begin{aligned}\mathcal{B}_4(t) = 2^{-11}\prod_{i<j}^{4}\frac{1}{\zeta_i^2-\zeta_j^2}\Big\{ & W(1,2,3,4)\cosh\tfrac{1}{2}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)\\ &+W(1,2,\bar{4},\bar{3})\cosh\tfrac{1}{2}(\varphi_1+\varphi_2-\varphi_3-\varphi_4)\\ &+W(1,3,\bar{4},\bar{2})\cosh\tfrac{1}{2}(\varphi_1-\varphi_2+\varphi_3-\varphi_4)\\ &+W(2,3,\bar{4},\bar{1})\cosh\tfrac{1}{2}(\varphi_1-\varphi_2-\varphi_3+\varphi_4)\Big\}\end{aligned} \tag{3.44d}$$

$$\varphi_k(t) = -2\zeta_k t + \varphi_{0k}. \tag{3.44e}$$

We also note that $\bar{4}=5, \bar{3}=6, \bar{2}=7, \bar{1}=8$; $W(i,j,k,l)$ *are defined by (3.4) and in both the summation and the products, denoted above by* $i<j$ *we mean that* i *and* j *take values from 1 to 4.*

It is well known that the automorphism v_1 maps not only the fundamental weights ω_k as in (3.41) but also the whole sets of weights, e.g. $v_1 : \Gamma(\omega_1) \to \Gamma(\omega_3) \to \Gamma(\omega_4)$. Using the definition of $\mathcal{B}_k(t)$ and the properties of the scalar products $(\vec{\zeta}, v_1\gamma^{(k)}) = (v_1^{-1}\vec{\zeta}, \gamma^{(k)})$ we obtain

$$\begin{aligned}&\mathcal{B}_1(t;\vec{\zeta},\vec{\varphi}_0) = \mathcal{B}_3(t; v_1^{-1}\vec{\zeta}, v_1^{-1}\vec{\varphi}_0) = \mathcal{B}_4(t; v_1\vec{\zeta}, v_1\vec{\varphi}_0)\\ &\mathcal{B}_2(t;\vec{\zeta},\vec{\varphi}_0) = \mathcal{B}_2(t; v_1\vec{\zeta}, v_1\vec{\varphi}_0).\end{aligned} \tag{3.45}$$

These relations are compatible with (1.3), i.e.

$$(\vec{q}(t), \omega_{3,4}) = \ln\mathcal{B}_{3,4}(t;\zeta,\varphi_0) = (\vec{q}(t), v_1^{\pm 1}\omega_1) = \ln\mathcal{B}_1(t; v_1^{\pm 1}\vec{\zeta}, v_1^{\pm 1}\vec{\varphi}_0).$$

Example 2. *Next we will derive the result for* $\boldsymbol{B}_3$ *starting from the* $\boldsymbol{D}_4$ *case. It is well known that* $\boldsymbol{B}_3$ *can be obtained from* $\boldsymbol{D}_4$ *by imposing a symmetry condition with respect to the outer-automorphism* v_2 *of* $\boldsymbol{D}_4$ *defined by*

$$v_2\alpha_k = \alpha_k \quad \textit{for } k=1,2 \qquad v_2\alpha_3 = \alpha_4 \qquad v_2\alpha_4 = \alpha_3. \tag{3.46}$$

The symmetry with respect to v_2 *reflects on the scattering data of the* $\boldsymbol{D}_4$ *case by*

$$\zeta_4 = 0 \qquad \varphi_{04} = 0 \qquad \textit{or} \qquad \varphi_4(t) = 0. \tag{3.47}$$

To the end of this example, when referring to the variables related to $\boldsymbol{D}_4$ *we will assume these conditions to be imposed and will denote this fact by an additional 'prime'; the corresponding* *variables for the $\boldsymbol{B}_3$ case will be denoted by the same letters with an additional 'tilde'. Then inserting (3.47) into (2.14) we can write*

$$w'_k = 2\tilde{w}_k \quad k = 1, 2, 3 \qquad w'_4 = \tilde{w}_4. \tag{3.48}$$

In analogy with (3.45) we have

$$\mathcal{B}'_4(t; \vec{\zeta}, \vec{\varphi}_0) = \mathcal{B}'_3(t; v_2\vec{\zeta}, v_2\vec{\varphi}_0) \tag{3.49}$$

and after imposing the v_2-involution symmetry, namely, $(\vec{q}(t), \omega_k) = (\vec{q}(t), v_2\omega_k)$ for each k we have $q'_4 = 0$. Indeed, using (3.47) one can find that $\mathcal{B}'_4(t; \vec{\zeta}, \vec{\varphi}_0) = \mathcal{B}'_3(t; \vec{\zeta}, \vec{\varphi}_0)$ which leads to $q'_4 = 0$. In this way q'_1, q'_2 and q'_3 give us solutions of a system, equivalent to that of $\boldsymbol{B}_3$. More precisely after some rearrangements we find that

$$\mathcal{B}'_1(t) = 2\tilde{\mathcal{B}}_1(t) \qquad \mathcal{B}'_2(t) = 4\tilde{\mathcal{B}}_2(t) \qquad \mathcal{B}'_3(t) = \mathcal{B}'_4(t) = 2^{3/2}\tilde{\mathcal{B}}_3(t). \tag{3.50}$$

Comparing (3.50) with (3.43) we see that

$$q'_k(t) = \tilde{q}_k(t) + \ln 2 \quad k = 1, 2, 3 \qquad q'_4(t) = 0 \tag{3.51}$$

where

$$\tilde{q}_1(t) = \ln \tilde{\mathcal{B}}_1(t) \qquad \tilde{q}_2(t) = \ln \frac{\tilde{\mathcal{B}}_2(t)}{\tilde{\mathcal{B}}_1(t)} \qquad \tilde{q}_3(t) = \ln \frac{\tilde{\mathcal{B}}_3^2(t)}{\tilde{\mathcal{B}}_2(t)}. \tag{3.52}$$

The condition (3.47), or equivalently, $q'_4(t) = 0$ when imposed onto the system of equations (3.42) leads to a system for $q'_k(t)$, $k = 1, 2, 3$ slightly different from the $\boldsymbol{B}_3$-CTC. The difference is in the coefficient in front of $\alpha_3 e^{-(\vec{q}', \alpha_3)}$ which comes out with an additional factor of 2. This factor is precisely cancelled if we go over to the variables $\tilde{q}_k(t)$.

Quite analogously, but with more technicalities, one can prove that the symmetry with respect to the outer-automorphism v_2 of $\boldsymbol{D}_r$ will reduce the $\boldsymbol{D}_r$ solution to that for $\boldsymbol{B}_{r-1}$; more precisely, using an analogous notation as above, we can write

$$\mathcal{B}'_k(t) = 2^k\tilde{\mathcal{B}}_k(t) \quad \text{for } k = 1\ldots, r-2 \qquad \mathcal{B}'_{r-1}(t) = \mathcal{B}'_r(t) = 2^{(r-1)/2}\tilde{\mathcal{B}}_{r-1}(t) \tag{3.53}$$

and

$$\begin{aligned} &q'_k(t) = \tilde{q}_k(t) + \ln 2 \quad \text{for } k = 1, \ldots, r-1 \qquad q'_r(t) = 0 \\ &\tilde{q}_k(t) = \ln \frac{\tilde{\mathcal{B}}_k(t)}{\tilde{\mathcal{B}}_{k-1}(t)} \quad \text{for } k = 1, \ldots, r-2 \qquad \tilde{q}_{r-1}(t) = \ln \frac{\tilde{\mathcal{B}}_{r-1}^2(t)}{\tilde{\mathcal{B}}_{r-2}(t)}. \end{aligned} \tag{3.54}$$

In deriving (3.50) we see, that due to $\zeta_4 = 0$ two of the terms in $\mathcal{B}_1(t)$ combine together; this corresponds to the fact that $\Gamma_{\boldsymbol{B}_3}(\omega_1)$ has only seven weights, while $\Gamma_{\boldsymbol{D}_4}(\omega_1)$ has eight. Analogous, but more complicated combinations and cancellations take place in the proof of (3.54).

Example 3. *The case with $\boldsymbol{G}_2$ can be obtained from $\boldsymbol{D}_4$ after imposing a symmetry condition with respect to the outer-automorphism v_1 (3.41) of $\boldsymbol{D}_4$ of order three. The restrictions that v_1 imposes on the scattering data are*

$$\zeta_4 = 0 \qquad \zeta_1 - \zeta_2 = \zeta_3 \qquad \varphi_{04} = 0 \qquad \varphi_{01} - \varphi_{02} = \varphi_{03} \tag{3.55}$$

or, in other words,

$$v_1(\vec{\zeta}) = \vec{\zeta} \qquad v_1(\vec{\varphi}_0) = \vec{\varphi}_0.$$

Then one can check that due to (3.45) $\vec{q}(t)$ is invariant with respect to v_1: $v_1(\vec{q}(t)) = \vec{q}(t)$ and, consequently, is an element of the subalgebra $\boldsymbol{G}_2$.

Indeed, if we average the set of simple roots of $\boldsymbol{D}_4$ with respect to the action of v_1 and also the system of equations (3.42) then we obtain the system of simple roots of $\boldsymbol{G}_2$:

$$\beta_1 = \tfrac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4) = \tfrac{1}{3}(e_1 - e_2 + 2e_3) \qquad \beta_2 = \alpha_2 = e_2 - e_3 \tag{3.56}$$

and the system of equations

$$Q_{1,tt} = \mathrm{e}^{-(\vec{Q},\beta_1)} \qquad Q_{2,tt} = \mathrm{e}^{-(\vec{Q},\beta_2)} - \mathrm{e}^{-(\vec{Q},\beta_1)} \tag{3.57}$$

which is slightly different from (1.1) for $\boldsymbol{G}_2$; namely the variables, let say q'_1, q'_2 and $q'_3 = -q'_1 - q'_2$, entering in the original system (1.1) for $\boldsymbol{G}_2$ are

$$q'_1(t) = Q_1(t) - 2\ln 3 \qquad q'_2 = Q_2 - \ln 3 \tag{3.58}$$

The solution to (3.57) is provided by

$$Q_1 = \ln \mathcal{B}_1(t; \vec{\zeta}, \vec{\varphi}_0) \qquad Q_2 = \ln \frac{\mathcal{B}_2(t; \vec{\zeta}, \vec{\varphi}_0)}{\mathcal{B}_1(t; \vec{\zeta}, \vec{\varphi}_0)} \tag{3.59}$$

where $\mathcal{B}_{1,2}(t)$ are obtained from (3.44a) and (3.44b) with the additional restrictions (3.55); namely, we have

$$\begin{aligned}
\mathcal{B}_1(t; \vec{\zeta}, \vec{\varphi}_0) &= 2b_0 b_1 \left[\frac{\zeta_2 - \zeta_3}{\zeta_1} \cosh\varphi_1 + \frac{\zeta_1 + \zeta_3}{\zeta_2} \cosh\varphi_2 + \frac{\zeta_1 + \zeta_2}{\zeta_3} \cosh\varphi_3 \right] + 2b_1^2 \\
\mathcal{B}_2(t; \vec{\zeta}, \vec{\varphi}_0) &= b_0 b_1^2 \left[\frac{\zeta_1^2}{\zeta_2\zeta_3} \frac{\cosh(\vec{01}, \vec{\varphi})}{\zeta_2 - \zeta_3} + \frac{\zeta_2^2}{\zeta_1\zeta_3} \frac{\cosh(\vec{31}, \vec{\varphi})}{\zeta_1 + \zeta_3} + \frac{\zeta_3^2}{\zeta_1\zeta_2} \frac{\cosh(\vec{32}, \vec{\varphi})}{\zeta_1 + \zeta_2} \right. \\
&\quad + \frac{3(\zeta_1 + \zeta_2)}{\zeta_1\zeta_2} \cosh(\vec{10}, \vec{\varphi}) + \frac{3(\zeta_1 + \zeta_3)}{\zeta_1\zeta_3} \cosh(\vec{11}, \vec{\varphi}) \\
&\quad \left. + \frac{3(\zeta_2 - \zeta_3)}{\zeta_2\zeta_3} \cosh(\vec{21}, \vec{\varphi}) \right] + 12 b_0^2 b_1 \left(\frac{1}{\zeta_3} + \frac{1}{\zeta_2} - \frac{1}{\zeta_1} \right) \\
b_0 &= \frac{1}{8(\zeta_1 + \zeta_2)(\zeta_1 + \zeta_3)(\zeta_2 - \zeta_3)} \qquad b_1 = \frac{1}{8\zeta_1\zeta_2\zeta_3}
\end{aligned} \tag{3.60}$$

where by $\vec{ij}$ we denote the root $i\beta_1 + j\beta_2$ of $\boldsymbol{G}_2$. Obviously each of the terms in $\mathcal{B}_k(t)$ in (3.60) can be related to a weight of the corresponding fundamental representation $\Gamma(\omega_k)$ of $\boldsymbol{G}_2$; so again we cast the solution in the form (1.3).

We summarize the results of this section by the following remark. One can view the solutions of the CTC related to the classical series $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ as being obtained from the Moser formulae combined with the corresponding constraint on the scattering data. The case of $\boldsymbol{D}_r$ can be obtained likewise if we take the Lax matrix to be pentadiagonal as in (A.8).

On the other hand, starting from a Lax matrix related to each of these series we can always apply Moser's approach and derive as a result the functions $q_k(t)$, $k = 1, \ldots, N$. In order for both answers for the CTC-solutions to be compatible one needs to show that

$$q_k(t) = -q_{\bar{k}}(t) \qquad \bar{k} = N + 1 - k \tag{3.61}$$

where N is the dimension of the typical representation of $\mathfrak{g}$. Equation (3.61) can be derived from the results in sections 2 and 3 and shows the compatibility of the two approaches to the CTC solutions.

4. Dynamical regimes and large time asymptotics

There are important differences between the RTC and CTC, especially in the asymptotic behaviour of their solutions. Indeed, for the RTC, one has [4, 12] that, both the eigenvalues, ζ_k and the constants $\varphi_k(0)$, are always real-valued. Moreover, one can prove that $\zeta_k \neq \zeta_j$ for $k \neq j$, i.e. no two eigenvalues can be exactly the same. As a direct consequence of this, it follows that the only possible asymptotic behaviour in the RTC is an asymptotically separating, free motion of the particles.

The situation is different for the CTC. Now the eigenvalues $\zeta_k = \kappa_k + \mathrm{i}\eta_k$, as well as the constants $\varphi_k(0)$ become complex. Furthermore, the argument of Moser [4] does not apply to the complex case, so one can have multiple eigenvalues. The collection of eigenvalues, ζ_k, still determines the asymptotic behaviour of the solutions. In particular, it is κ_k that determines the asymptotic velocity of the kth particle. For simplicity, we assume $\zeta_k \neq \zeta_j$ for $k \neq j$. However, this condition does not necessarily mean that $\kappa_k \neq \kappa_j$. We also assume that the κ_k's are ordered as

$$\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_N. \tag{4.1}$$

This ordering is known as the sorting condition. More generally it can be understood as $-\vec{\kappa} \in \bar{W}_D$—the closure of the dominant Weyl chamber. Once this is done, for the corresponding set of N particles there are three possible general configurations.

(A) Free-particle propagation (Moser case); then $q_k(t)$ have a linear-in-t asymptotic behaviour and $-\vec{\kappa}$ is in the interior of W_D.

(B) Bound state(s) and mixed regimes when one (or several) group(s) of particles form a bound state; then each group of particles oscillate around a common trajectory with a linear-in-t asymptotic behaviour; then $-(\vec{\kappa}, \alpha_k) = 0$ for some set of indices $k \in I_{\mathrm{bs}}$.

(C) Degenerate solutions when two (or more) of the eigenvalues $\zeta_k = \zeta_{k+1} = \cdots$ are equal, then $q_k(t) - q_{k+1}(t)$ have a logarithmic-in-t asymptotic behaviour, i.e. the distance between the particles grows as $\ln t$.

Obviously cases (B) and (C) have no analogues in the RTC and physically are qualitatively different from (A).

4.1. Asymptotically free regimes

We begin with the first possibility from the above-mentioned free-particle asymptotics (the Moser case). It is realized that if we require that all real parts of the eigenvalues κ_k are pairwise different; i.e. $-\vec{\kappa}$ belongs to the interior of the dominant Weyl chamber W_D:

$$(-\vec{\kappa}, \alpha_s) > 0 \qquad s = 1, \ldots, r \tag{4.2}$$

while the imaginary parts η_k may be arbitrary.

Let us now consider the asymptotics for the $\boldsymbol{A}_r$, $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{D}_r$ series. Using the explicit expressions for $\mathcal{B}_k(t)$ it is not difficult to evaluate their asymptotic behaviour for $t \to \pm\infty$:

$$\mathcal{B}^{\pm}_{k,\mathrm{as}}(t) = W^{(k)}(\vec{\zeta}, \omega_k^{\pm})\, \mathrm{e}^{(-2\vec{\zeta}t + \vec{\varphi}_0, \omega_k^{\pm})}\big(1 + \mathcal{O}(\exp(\mp K_k^{\pm} t))\big). \tag{4.3}$$

Here $\omega_k^{\pm}$ are the highest (lowest) weights, related through the Weyl group element w_0: $\omega_k^+ = w_0(\omega_k^-)$ and

$$K_k^+ = \min_{\gamma\in\Gamma(\omega_k^+)\backslash\omega_k^+} \mathrm{Re}(\vec{\zeta}, \gamma - \omega_k^+) = -(\vec{\kappa}, \alpha_k)$$

$$K_k^- = \min_{\gamma\in\Gamma(\omega_k^+)\backslash\omega_k^-} \mathrm{Re}(\vec{\zeta}, \gamma - \omega_k^-) = -(\vec{\kappa}, \alpha_{\bar{k}})$$

see appendix B. (4.4)

Note the natural way in which the two asymptotics are related by the w_0 transformation of the Weyl group, namely

$$\mathcal{B}_{k,\mathrm{as}}^-(t) = w_0\left(\mathcal{B}_{k,\mathrm{as}}^+(t)\right) = W^{(k)}(\vec{\zeta}, w_0(\omega_k^+))\,\mathrm{e}^{(\vec{\varphi}(t), w_0(\omega_k^+))}$$
$$= W^{(k)}(w_0(\vec{\zeta}), \omega_k^+)\,\mathrm{e}^{(w_0(\vec{\varphi}(t)), \omega_k^+)}. \tag{4.5}$$

More generally since $W^{(k)}(\vec{\zeta}, \gamma)$ depends only on the scalar products of the type $(\vec{\zeta}, \gamma)$ the action of the automorphism w_0 on $W^{(k)}(\vec{\zeta}, \gamma)$ is given by $W^{(k)}(\vec{\zeta}, w_0(\gamma)) = W^{(k)}(w_0(\vec{\zeta}), \gamma)$.

The relations (4.3) and (4.4) are due to the simple fact that the leading exponent for $t \to \infty$ ($t \to -\infty$) in $\mathcal{B}_k(t)$ corresponds to the weight $\gamma \in \Gamma(\omega_k)$ for which the value of $\mathrm{Re}(-\vec{\zeta}, \gamma)$ is maximal (minimal). Since $-\vec{\kappa} \in W_D$ this maximum (minimum) is realized when $\gamma = \omega_k^+$ (respectively, $\gamma = \omega_k^-$).

From the previous considerations we have

$$q_k(t) = \ln \frac{B_k(t)}{B_{k-1}(t)} = \sum_{s=1}^{r} \frac{2(\alpha_s, e_k)}{(\alpha_s, \alpha_s)} \ln \mathcal{B}_s(t) \tag{4.6}$$

and consequently the asymptotics $\vec{q}_{\mathrm{as}}^{\,\pm}$ of $\vec{q}(t)$ for $t \to \pm\infty$ are given by

$$\vec{q}_{\mathrm{as}}^{\,+}(t) = -2\vec{\zeta}t + \vec{\varphi}_0 + \sum_{k=1}^{r} \frac{2\alpha_k}{(\alpha_k, \alpha_k)} \ln W^{(k)}(\vec{\zeta}, \omega_k^+) \tag{4.7}$$

$$\vec{q}_{\mathrm{as}}^{\,-}(t) = w_0\big(-2\vec{\zeta}t + \vec{\varphi}_0\big) + \sum_{k=1}^{r} \frac{2\alpha_k}{(\alpha_k, \alpha_k)} \ln W^{(k)}(w_0(\vec{\zeta}), \omega_k^+) \tag{4.8}$$

up to terms falling off exponentially for $t \to \pm\infty$. The explicit expressions for the components $q_k(t)$ for the $\boldsymbol{A}_r$ series are well known, see, e.g., [4, 12, 20]. For the other classical series of Lie algebras we obtain

$$q_{k,\mathrm{as}}^{\pm}(t) = \mp 2\zeta_k t \pm \varphi_{0k} + \beta_k + \mathcal{O}\big(\mathrm{e}^{\mp N_k^{\pm} t}\big) \qquad k = 1, \ldots, r \tag{4.9}$$

with

$$\beta_k = \ln\left(w_k \prod_{s=1}^{k-1} (2\zeta_s - \zeta_k)^2\right) \qquad N_k^{\pm} = \min_{s:(e_k,\alpha_s)\neq 0} K_s^{\pm}. \tag{4.10}$$

The only exception to (4.9) and (4.10) is for $\mathfrak{g} \simeq \boldsymbol{D}_r$ with odd r, $k = r$ and $t \to -\infty$; then

$$q_{r,\mathrm{as}}^-(t) = -2\zeta_r t + \varphi_{0r} + \ln\left(w_r \prod_{s=1}^{r-1} (2\zeta_r + 2\zeta_s)^2\right). \tag{4.11}$$

It has been known for a long time [15, 16] that for the RTC the asymptotic velocities $\vec{v}^{\,\pm}$ are related by $\vec{v}^{\,-} = w_0(\vec{v}^{\,+})$. In the complex case the analogues of $v^{\pm}$ are the complex vectors $-2\vec{\zeta}$ and $-2w_0(\vec{\zeta})$, respectively.

Up to now we know of only one physical application of CTC as a model describing the N-soliton train interactions [17–20]. Gaining insight from it we will interpret $\mathrm{Re}\, q_k(t)$ as the trajectory of the centre of mass of the kth 'particle' (soliton). Besides each particle is complex and possesses an internal degree of freedom. Then $-2\,\mathrm{Re}\,\zeta_k = -2\kappa_k$ will be the asymptotic velocity of the kth particle at $t \to \infty$, while $-2\,\mathrm{Im}\,\zeta_k = -2\eta_k$ determines its asymptotic phase velocity.

4.2. Mixed regimes for $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{D}_{2n}$

Our aim in this and the next subsection will be to consider the cases when two or more particles form bound state(s); we will say that several particles form a bound state if they have equal asymptotic velocities. In this subsection we consider only those members of the classical series for which $w_0 \equiv -\mathrm{id}$.

Bound state(s) are possible when $-\vec{\kappa}$ is on the boundaries of W_D, i.e. if we have

$$-(\alpha_k, \vec{\kappa}) = 0 \qquad k \in I_{\mathrm{bs}} \tag{4.12}$$

where $I_{\mathrm{bs}} \subset \{1, \ldots, r\}$ is a subset of indices. If $I_{\mathrm{bs}} = \{m\}$ contains just one index $m < r$ then $\kappa_m = \kappa_{m+1}$ and we will have a two-particle bound state. If $I_{\mathrm{bs}} = \{m, m+1, \ldots, m+p\}$ and $m + p < r$ then $\kappa_m = \kappa_{m+1} = \cdots = \kappa_{m+p}$ and we have a $(p+1)$-particle bound state. The cases when the largest index in I_{bs} is equal to r should be considered separately; indeed, due to the fact that the sets of simple roots for the different classical series differ only in the choices for α_r these cases may lead to substantially different results.

In our previous paper [20] we obtained the large-time asymptotics for the two-particle bound states in the $\boldsymbol{A}_r$ CTC. Here we will briefly analyse more general cases when:

(a) $\mathfrak{g}$ belongs to the other classical series and one bound state may be present, i.e. when $I_{\mathrm{bs}} = \{m\}$, $m \leqslant r$ and $I_{\mathrm{bs}} = \{m, m+1\}$, $m+1 \leqslant r$;
(b) two bound states may be present, i.e. $I_{\mathrm{bs}} = \{m, p\}$, $m + 1 < p \leqslant r$.

For brevity we will write down the asymptotics only for those components $q_k(t)$ which differ from the typical ones (4.9). We will limit ourselves to the cases when the mixed regime contains two- and three-particle bound states only. The other more complicated regimes can be analysed analogously.

Indeed, if for $k \in I_{\mathrm{bs}}$ we have $(\vec{\kappa}, \alpha_k) = 0$ then at least two terms in $\mathcal{B}_k(t)$ may have the same asymptotic behaviour. To our purpose it is sufficient to evaluate only the leading exponents. Thus we find

$$\mathcal{B}^{\pm}_{p,\mathrm{as}}(t) = \mathrm{e}^{(\varphi(t),\omega_p^{\pm})}\left[W^{(p)}(\vec{\zeta}, \omega_p^{\pm}) + \sum_{\alpha \in G_p^{\pm}(\vec{\kappa})} \bar{W}^{(p)}(\alpha) + \mathcal{O}\big(\mathrm{e}^{\mp K_p'^{,\pm} t}\big)\right] \tag{4.13}$$

where

$$K_p'^{,\pm} = \min_{\gamma \in \Gamma_{p,\pm}(\omega_p^+)} \left[-2(\vec{\kappa}, \omega_p^+ - \gamma)\right] \qquad \bar{W}^{(p)}(\alpha) = \mathrm{e}^{-(\varphi(t),\alpha)} W^{(p)}(\vec{\zeta}, \omega_p^{\pm} \mp \alpha)$$

$$G_p^{\pm}(\vec{\kappa}) = \left\{\alpha > 0,\ (\alpha, \vec{\kappa}) = 0,\ \pm\frac{2(\alpha, \omega_p^{\pm})}{(\alpha, \alpha)} \geqslant 1\right\} \tag{4.14}$$

$$\Gamma_{p,\pm}(\omega_p^+) = \Gamma(\omega_p^+) \backslash \left\{\omega_p^{\pm}, \omega_p^{\pm} \mp \alpha, \alpha \in G_p^{\pm}(\vec{\kappa})\right\}$$

The condition $(\alpha, \vec{\kappa}) = 0$ ensures that $\bar{W}^{(p)}$ only oscillates when $t \to \pm\infty$, while the third condition in $G_p^{\pm}(\vec{\kappa})$ means that $\omega_p^{\pm} \mp \alpha \in \Gamma(\omega_p^+)$.

We start with the simplest case $I_{\mathrm{bs}} = \{m\}$, which contains several qualitatively different subcases which will be listed below. In each of them it is possible to describe the sets of roots $G_p^{\pm}(\vec{\kappa})$ and to evaluate the estimating exponents $K_p'^{\pm}$, for details see appendix B. It turns out that $G_p^{\pm}(\vec{\kappa}) \equiv \emptyset$ for $p \neq m$ and $G_m^{+}(\vec{\kappa}) \equiv \{\alpha_m\}$, $G_m^{-}(\vec{\kappa}) \equiv G_m^{+}(w_0(\vec{\kappa})) \equiv \{w_0(\alpha_m)\}$. In what follows we will concentrate mainly on the asymptotics for $t \to \infty$; the asymptotics for $t \to -\infty$ we will obtain by formula (4.8). Therefore, the asymptotics of $\mathcal{B}^{+}_{p,\mathrm{as}}(t)$ for $p \neq m$ will be given by (4.3), while for $\mathcal{B}^{+}_{m,\mathrm{as}}(t)$ we obtain

$$\mathcal{B}^{+}_{m,\mathrm{as}}(t) = \mathrm{e}^{(\varphi(t),\omega_m^{+})}\left[W^{(m)}(\vec{\zeta}, \omega_m^{+}) + \bar{W}^{(m)}(\alpha_m) + \mathcal{O}\left(\mathrm{e}^{-K_m'^{+}t}\right)\right] \tag{4.15}$$

with the following result for $K_m'^{+}$ valid for any of the algebras in the classical series (see appendix B):

$$K_m'^{+} = \min_{s:(\alpha_s,\alpha_m)\neq 0}\left[-2(\vec{\kappa}, \alpha_s)\right]. \tag{4.16}$$

In other words, the minimum should be taken with respect to the simple roots α_s that are connected to α_m in the Dynkin diagram.

Now we are in a position to compare the asymptotic velocities of the particles and to single out the structure of the bound states (if any). Since the asymptotic velocity of $q_k(t)$ for $t \to \infty$ is equal to $-2\kappa_k$ we just have to see what constraints on $\{\kappa_1, \ldots, \kappa_r\}$ will be imposed by $-(\vec{\kappa}, \alpha_k) > 0$, $k \neq m$ and $-(\vec{\kappa}, \alpha_m) = 0$. For $\mathfrak{g} \simeq \boldsymbol{B}_r, \boldsymbol{C}_r$ and $m < r$ we have

$$\kappa_1 < \cdots < \kappa_m = \kappa_{m+1} < \cdots < \kappa_r < 0 \tag{4.17}$$

for $\mathfrak{g} \simeq \boldsymbol{D}_{2n}$ and $m < 2n-1$, $m = 2n-1$ we have

$$\kappa_1 < \cdots < \kappa_m = \kappa_{m+1} < \cdots < \kappa_{2n-1} < -|\kappa_{2n}| < 0 \tag{4.18}$$

and

$$\kappa_1 < \cdots < \kappa_m < \kappa_{m+1} < \cdots < \kappa_{2n-1} = \kappa_{2n} < 0 \tag{4.19}$$

respectively.

Finally, for $\mathfrak{g} \simeq \boldsymbol{B}_r, \boldsymbol{C}_r$ and $m = r$ we obtain

$$\kappa_1 < \cdots < \kappa_{r-1} < \kappa_r = 0 \tag{4.20}$$

and for $\mathfrak{g} \simeq \boldsymbol{D}_{2n}$ and $m = 2n$

$$\kappa_1 < \cdots < \kappa_{2n-1} = -\kappa_{2n} < 0. \tag{4.21}$$

From (4.17)–(4.19) it is easy to see that for $m < r$ we always have one bound state of two particles (mth and $(m+1)$st); the rest of the particles go into a free asymptotic regime. If in addition $w_0 = -\mathrm{id}$, as we assumed in the beginning of this subsection, this bound state will also be present for $t \to -\infty$. Therefore, for this class of algebras we have stable two-particle bound states for all $m < r$.

For $m = r$ the situation is different. From (4.20) we see that the condition $-(\vec{\kappa}, \alpha_r) = 0$ for $\mathfrak{g} \simeq \boldsymbol{B}_r, \boldsymbol{C}_r$ just means that the rth particle has vanishing velocity. As for $\mathfrak{g} \simeq \boldsymbol{D}_{2n}$ the condition $-(\vec{\kappa}, \alpha_{2n}) = 0$ means that the $(2n-1)$st and the $2n$th particles have opposite velocities. Therefore, for $m = 2n$ no bound states are possible.

The next possibility is that the set $I_{\mathrm{bs}} \equiv \{m, p\}$. There are qualitatively different cases here: (a) $(\alpha_m, \alpha_p) = 0$ and (b) $(\alpha_m, \alpha_p) \neq 0$. Each of the values m and p in case (a) can be considered independently and to each of them applies the analysis already posed above. In the generic case $m < p < r$ we will have two pairs of bound states each containing two particles; if $m < p = r$ then we have only one bound state of two particles. An exception here is the case $\mathfrak{g} \simeq \boldsymbol{D}_{2n}$ and $m = 2n - 1$, $p = 2n$. The two roots α_{2n-1} and α_{2n} are obviously orthogonal, but now the condition (4.12) leads to $\kappa_{2n} = \kappa_{2n-1} = 0$ and as a result in this case we have only one bound state consisting of two particles with vanishing velocities.

Let us now analyse case (b). For generic values of $m < p < r$ what we find is a bound state of three particles. One possible realization of (b) is to take $p = m + 1 < r$; then the condition (4.12) leads to

$$\kappa_1 < \cdots < \kappa_m = \kappa_{m+1} = \kappa_{m+2} < \cdots \kappa_{m+3} < \cdots \tag{4.22}$$

i.e. the particles numbered by m, $m + 1$ and $m + 2$ move with the same asymptotic velocities and form a bound state. Again we must look through all possibilities when case (b) takes place and point out possible exceptions; such as, for example, the case when $\mathfrak{g} \simeq \boldsymbol{B}_r, \boldsymbol{C}_r$ and $m = r - 1$, $p = r$. Equation (4.12) then gives $\kappa_{r-1} = \kappa_r = 0$, which means that this is a bound state of two particles: the $(r - 1)$st and the rth with vanishing velocity.

If $\mathfrak{g} \simeq \boldsymbol{D}_{2n}$ and $m = 2n - 2$, $p = 2n - 1$ we obtain a three-particle bound state with velocity $\kappa_{2n-2} = \kappa_{2n-1} = \kappa_{2n} < 0$. The last example related to this algebra is $m = 2n - 2$, $p = 2n$ which corresponds to $\kappa_{2n-2} = \kappa_{2n-1} = -\kappa_{2n}$. This means that the particles $2n - 2$ and $2n - 1$ form a bound state, but the last $2n$th particle moves with the opposite velocity and is not part of the bound state.

Obviously the number of examples can be extended to include sets $I_{\rm bs}$ with more indices; one can expect to have bound states with an increasing number of bounded particles. It is not difficult to also present the explicit form of the asymptotics of $q^{\pm}_{k,\rm as}(t)$. The most difficult part in this calculation is to determine the sets of roots $G^{\pm}_p(\vec{\kappa})$. We list these sets of roots in appendix B for the classical series of Lie algebras related to the sets $I_{\rm bs}$ with one and two indices. Indeed, if we choose $\mathfrak{g} \simeq \boldsymbol{B}_r$, $I_{\rm bs} = \{r - 1, r\}$. Then the sets of roots

$$G^{+}_{r-1}(\vec{\kappa}) = \{\alpha_{r-1}, \alpha_{r-1} + \alpha_r, \alpha_{r-1} + 2\alpha_r\} \qquad G^{+}_{r}(\vec{\kappa}) = \{\alpha_r, \alpha_{r-1} + \alpha_r, \alpha_{r-1} + 2\alpha_r\}.$$

Then

$$\begin{aligned}
\mathcal{B}^{+}_{r-1,\rm as} &= \mathrm{e}^{(\varphi(t),\omega^{+}_{r-1})}\big[W^{(r-1)}(\vec{\zeta}, \omega^{+}_{r-1}) + \bar{W}^{(r-1)}(\alpha_{r-1}) \\
&\qquad + \bar{W}^{(r-1)}(\alpha_{r-1} + \alpha_r) + \bar{W}^{(r-1)}(\alpha_{r-1} + 2\alpha_r)\big] \\
\mathcal{B}^{+}_{r,\rm as} &= \mathrm{e}^{(\varphi(t),\omega^{+}_{r})}\big[W^{(r)}(\vec{\zeta}, \omega^{+}_{r}) + \bar{W}^{(r)}(\alpha_r) + \bar{W}^{(r)}(\alpha_{r-1} + \alpha_r) + \bar{W}^{(r)}(\alpha_{r-1} + 2\alpha_r)\big].
\end{aligned} \tag{4.23}$$

Now we have to insert (4.23) into (4.6). After some calculations we obtain $\kappa_{r-1} = \kappa_r = 0$ and

$$\begin{aligned}
q^{+}_{r-1,\rm as}(t) &= -2\mathrm{i}\eta_{r-1}t + \varphi_{0r-1} + \ln \frac{\Sigma_{r-1}}{W^{(r-2)}(\vec{\zeta}, \omega^{+}_{r-2})} \\
q^{+}_{r,\rm as}(t) &= -2\mathrm{i}\eta_r t + \varphi_{0r} + \ln \frac{\Sigma_r^2}{\Sigma_{r-1}} \\
\Sigma_p &= W^{(p)}(\vec{\zeta}, \omega^{+}_p) + \tilde{\Sigma}_p \qquad p = r - 1, r \\
\tilde{\Sigma}_p &= \bar{W}^{(p)}(\alpha_p) + \bar{W}^{(p)}(\alpha_{r-1} + \alpha_r) + \bar{W}^{(p)}(\alpha_{r-1} + 2\alpha_r).
\end{aligned} \tag{4.24}$$

In this subsection we have presented various types of mixed regimes which could be called regular. By regular here we mean that the number and the structure of the bound states at $t \to -\infty$ coincides with that for $t \to \infty$. In the next subsection we consider the 'irregular' mixed regimes, which change qualitatively their structure during the evolution.

4.3. Mixed regimes for $\boldsymbol{D}_{2n+1}$: 'creation' and 'decay' of bound states

The 'irregular' mixed regimes take place only for algebras for which $w_0 \neq -\mathrm{id}$. This takes place for $\mathfrak{g} \simeq \boldsymbol{A}_r$ and $\mathfrak{g} \simeq \boldsymbol{D}_{2n+1}$. At the end of this subsection we will explain why the 'irregular' regimes, i.e. the effects of 'creation' and 'decay' of bound states can be related only to $\mathfrak{g} \simeq \boldsymbol{D}_{2n+1}$.

First of all we note that most of the bound states related to $\boldsymbol{D}_{2n+1}$ are regular. So are the states corresponding to $I_{\mathrm{bs}} = \{m\}$ with $m < 2n$ and $I_{\mathrm{bs}} = \{m, p\}$ with $m < p < 2n$. The formulae for the asymptotics in all these cases are quite analogous to the ones already presented.

Let us start with the first 'irregular' case with $I_{\mathrm{bs}} = \{2n\}$. This means that $\kappa_{2n} = \kappa_{2n+1}$ and for $t \to \pm\infty$

$$
\begin{aligned}
q^{\pm}_{2n,\mathrm{as}} &= \mp 2\kappa_{2n} t \mp 2\mathrm{i}\eta_{2n} t \pm \varphi_{0,2n} + \beta^{\pm,\prime}_{2n}(\vec{\zeta}) \\
q^{\pm}_{2n+1,\mathrm{as}} &= -2\kappa_{2n} t - 2\mathrm{i}\eta_{2n+1} t + \varphi_{0,2n+1} + \beta^{\pm,\prime}_{2n+1}(\vec{\zeta}) \\
\beta^{+,\prime}_{2n}(\vec{\zeta}) &= \ln \frac{\left[W^{(2n)}(\vec{\zeta}, \omega^{+}_{2n}) + \bar{W}^{(2n)}(\alpha_{2n})\right] W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1})}{W^{(2n-1)}(\vec{\zeta}, \omega^{+}_{2n-1})} \\
\beta^{+,\prime}_{2n+1}(\vec{\zeta}) &= \ln \frac{W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1})}{W^{(2n)}(\vec{\zeta}, \omega^{+}_{2n}) + \bar{W}^{(2n)}(\alpha_{2n})}
\end{aligned}
\tag{4.25}
$$

and $\beta^{-,\prime}_{k}(\vec{\zeta})$, $k = 2n, 2n+1$ are obtained from $\beta^{+,\prime}_{k}(\vec{\zeta})$ by using (1.5). Obviously at $t \to -\infty$ the $2n$th and the $(2n+1)$st particles have opposite velocities, while for $t \to \infty$ their velocities become equal. This situation can be viewed as 'creation' of a bound state.

The second 'irregular' case is with $I_{\mathrm{bs}} = \{2n+1\}$. This means that $\kappa_{2n} = -\kappa_{2n+1}$ and for $t \to \pm\infty$

$$
\begin{aligned}
q^{\pm}_{2n,\mathrm{as}} &= \mp 2\kappa_{2n} t \mp 2\mathrm{i}\eta_{2n} t \pm \varphi_{0,2n} + \beta^{\pm,\prime\prime}_{2n}(\vec{\zeta}) \\
q^{\pm}_{2n+1,\mathrm{as}} &= 2\kappa_{2n} t - 2\mathrm{i}\eta_{2n+1} t + \varphi_{0,2n+1} + \beta^{\pm,\prime}_{2n+1}(\vec{\zeta}) \\
\beta^{+,\prime\prime}_{2n}(\vec{\zeta}) &= \ln \frac{\left[W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1}) + \bar{W}^{(2n+1)}(\alpha_{2n+1})\right] W^{(2n)}(\vec{\zeta}, \omega^{+}_{2n})}{W^{(2n-1)}(\vec{\zeta}, \omega^{+}_{2n-1})} \\
\beta^{+,\prime\prime}_{2n+1}(\vec{\zeta}) &= \ln \frac{W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1}) + \bar{W}^{(2n+1)}(\alpha_{2n+1})}{W^{(2n)}(\vec{\zeta}, \omega^{+}_{2n})}
\end{aligned}
\tag{4.26}
$$

and again $\beta^{-,\prime}_{k}(\vec{\zeta})$, $k = 2n, 2n+1$ are obtained from $\beta^{+,\prime}_{k}(\vec{\zeta})$ by using (1.5). Now at $t \to -\infty$ the $2n$th and the $(2n+1)$st particles have equal velocities, while for $t \to \infty$ their velocities become opposite. This situation can be viewed as 'decay' of a bound state.

The next more complex situation is when $I_{\mathrm{bs}} = \{m, p\}$. Again we should consider two distinct subcases, namely $(\alpha_p, \alpha_m) = 0$ and $(\alpha_m, \alpha_p) \neq 0$.

In both cases we recover 'regular' asymptotics provided $m < p < 2n$; namely for such choices of I_{bs} we have either two pairs of two-particle bound states (if $(\alpha_m, \alpha_p) = 0$) or a three-particle bound state (if $(\alpha_m, \alpha_p) \neq 0$).

The 'irregular' cases with $(\alpha_m, \alpha_p) = 0$ are of two types. The first one takes place if $m < 2n$ and $p = 2n$ ($p = 2n+1$). Then for $t \to -\infty$ ($t \to \infty$) we have two bound states formed by the particles $\{m, m+1\}$ and $\{2n, 2n+1\}$, while at $t \to \infty$ ($t \to -\infty$) the second bound state decays and we are left with only one bound state. Quite different is the situation when $I_{\mathrm{bs}} = \{2n, 2n+1\}$. This corresponds to $\kappa_{2n} = \kappa_{2n+1} = 0$, so this is a regular case but with only *one bound state* formed by the particles $\{2n, 2n+1\}$ with vanishing velocity.

There are only two 'irregular' cases with $(\alpha_m, \alpha_p) \neq 0$, namely $I_{\rm bs} = \{2n-1, 2n\}$ and $I_{\rm bs} = \{2n-1, 2n+1\}$. The first one leads to $\kappa_{2n-1} = \kappa_{2n} = \kappa_{2n+1} < 0$ and to the following asymptotic behaviour of $q_k(t)$, $k = 2n-1$, $2n$ and $2n+1$:

$$\begin{aligned}
&q^{\pm}_{2n-1,\rm as} = \mp 2\kappa_{2n-1}t \mp 2{\rm i}\eta_{2n-1}t \pm \varphi_{0,2n-1} + \beta'^{,\pm}_{2n-1} \\
&q^{\pm}_{2n,\rm as} = \mp 2\kappa_{2n-1}t \mp 2{\rm i}\eta_{2n}t \pm \varphi_{0,2n} + \beta'^{,\pm}_{2n} \\
&q^{\pm}_{2n+1,\rm as} = -2\kappa_{2n-1}t - 2{\rm i}\eta_{2n+1}t + \varphi_{0,2n+1} + \beta'^{,\pm}_{2n+1} \\
&\beta'^{,+}_{2n-1} = \ln \frac{\Sigma'^{,+}_{2n-1}(\vec{\zeta})}{W^{(2n-2)}(\vec{\zeta}, \omega^{+}_{2n-2})} \qquad \beta'^{,+}_{2n} = \ln \frac{\Sigma'^{,+}_{2n}(\vec{\zeta})W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1})}{\Sigma'^{,+}_{2n-1}(\vec{\zeta})} \\
&\beta'^{,+}_{2n+1} = \ln \frac{W^{(2n+1)}(\vec{\zeta}, \omega^{+}_{2n+1})}{\Sigma'^{,+}_{2n}(\vec{\zeta})} \\
&\Sigma'^{,+}_{p}(\vec{\zeta}) = W^{(p)}(\vec{\zeta}, \omega^{+}_{p}) + \bar{W}^{(p)}(\alpha_p) + \bar{W}^{(p)}(\alpha_{2n-1} + \alpha_{2n}).
\end{aligned} \tag{4.27}$$

From these formulae we find that for $t \to -\infty$ we have a two-particle bound state formed by $(2n-1)$st and $2n$th particles, while for $t \to \infty$ the $(2n+1)$st particle 'joins' them and we have a three-particle bound state.

The case with $I_{\rm bs} = \{2n-1, 2n+1\}$ is analogous: the only difference is that at $t \to -\infty$ we have a three-particle bound state formed by the $(2n-1)$st, $2n$th and $(2n+1)$st particles, while for $t \to \infty$ the $(2n+1)$st particle 'separates' from them and we are left with a two-particle bound state.

Let us analyse this situation on the basis of our remark at the end of section 3. Let us first explain why such an irregular solution is not possible for the $\boldsymbol{A}_r$ series. In this case we have $r+1$ particles and the sets of asymptotic velocities for $t \to \infty$ and $t \to -\infty$ differ only in the ordering: $\{-2\kappa_{r+1} \leqslant -2\kappa_r \leqslant \cdots \leqslant -2\kappa_1\}$ and $\{-2\kappa_1 \geqslant -2\kappa_2 \geqslant \cdots \geqslant -2\kappa_{r+1}\}$, respectively. That is why it was quite natural to identify the kth particle at $t \to -\infty$ with the $\bar{k}$th particle at $t \to \infty$: they move with equal velocities. This is compatible with the action of w_0 in the $\boldsymbol{A}_r$ case, see (B.2). As a result if we have, say two bound states at $t \to -\infty$, i.e. $-2\kappa_1 = -2\kappa_2 > -2\kappa_3 = -2\kappa_4$ at $t \to \infty$ we will have again two bound states $-2\kappa_{r+1} = -2\kappa_r < -2\kappa_{r-1} = -2\kappa_{r-2}$.

Next we can view the solutions of CTC related to the classical series $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{D}_r$ as special symmetric solutions of the $sl(N)$-CTC, see (3.61). Then it is enough to consider only 'half' of the trajectories; the other half being obtained as a 'mirror' image. In this situation the sets of initial and final velocities are different and the identification, good for $\boldsymbol{A}_r$, is not possible for the other classical series; also quite different is the action of w_0 on the orthonormal basis of the root space, see (B.2).

As we mentioned above, if we consider the whole picture with all N trajectories we will see that no 'creation' or 'decay' of bound states takes place. In the cases of $\boldsymbol{D}_{2n+1}$ and $(\vec{\kappa}, \alpha_{2n}) = 0$ (or $(\vec{\kappa}, \alpha_{2n+1}) = 0$) this can be explained as follows. For $t \to \infty$ ($t \to -\infty$) we have a bound state between the $2n$th and $(2n+1)$st particle which for $t \to -\infty$ ($t \to \infty$) transfers into a bound state between the $2n$th and its 'mirror' symmetric $(2n+2)$th particle. 'Cutting' off the symmetric trajectories with numbers $2n+2, \ldots, N = 4n+2$ we find the effects described above.

Analogously we can explain the situation with a three-particle bound state at $t \to \pm\infty$ and a two-particle bound state at $t \to \mp\infty$. The whole picture of $4n+2$ particles will always contain a three-particle bound state.

4.4. Bound state regimes. Periodic and singular regimes

These regimes take place if $\vec{\kappa} = 0$, i.e. the set of eigenvalues $\zeta_k = \mathrm{i}\eta_k$ are purely imaginary. Then each of the functions $B_k(t)$ will be generically bounded. In particular, this means that all the complex 'particles' (or solitons) will move together, forming a bound state with a large number of degrees of freedom. In order to avoid degeneracies we have to request that $\vec{\eta} \in W_D$.

In order to have periodic solutions we need one more restrictions upon η_k, namely

$$\eta_k - \eta_m = s_{km}\eta_0 \tag{4.28}$$

where s_{km} are integers; if s_{km} are rational we can always make them integers by rescaling η_0.

Example 4. *Let $\mathfrak{g} \simeq \boldsymbol{B}_2 = so(5)$. The corresponding equations have the form*

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} = \mathrm{e}^{q_2 - q_1} \qquad \frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} = -\mathrm{e}^{q_2 - q_1} + \mathrm{e}^{-q_2} \tag{4.29}$$

and their periodic solutions are given by

$$q_1 = \ln \mathcal{B}_1(t) \qquad q_2 = \ln \frac{\mathcal{B}_2^2(t)}{\mathcal{B}_1(t)} \tag{4.30}$$

where

$$\mathcal{B}_1(t) = \frac{1}{16\eta_0^4(p_1^2 - p_2^2)} \left\{ \frac{\cos 2p_1(\Phi(t) + \Gamma)}{p_1^2} + \frac{\cos 2p_2(\Phi(t) - \Gamma)}{p_2^2} + \frac{p_1^2 - p_2^2}{p_1^2 p_2^2} \right\} \tag{4.31a}$$

$$\mathcal{B}_2(t) = \frac{-\mathrm{i}}{8\eta_0^3 p_1 p_2} \left\{ \frac{\cos(p_+\Phi(t) + p_-\Gamma)}{p_+} + \frac{\cos(p_-\Phi(t) + p_+\Gamma)}{p_-} \right\} \tag{4.31b}$$

where $\eta_k = p_k\eta_0$, p_k are integers and

$$\Phi(t) = \eta_0 t + \frac{\mathrm{i}}{4}\left(\frac{\phi_{01}}{p_1} + \frac{\phi_{02}}{p_2}\right) \qquad \Gamma = \frac{\mathrm{i}}{4}\left(\frac{\phi_{01}}{p_1} - \frac{\phi_{02}}{p_2}\right) \qquad p_\pm = p_1 \pm p_2. \tag{4.32}$$

The period is provided by

$$\tau = \frac{\pi}{\eta_0 s_0} \tag{4.33}$$

where s_0 is the greatest common divisor of p_1, p_2, p_+ and p_-.

Our next remark is that in the generic case when $\mathrm{Re}\,\phi_{0k} \neq 0$ the solution (4.31) is a regular one; then $|\mathcal{B}_1(t)|$ and $|\mathcal{B}_2(t)|$ are strictly positive for all t. If, however, we choose $\mathrm{Re}\,\phi_{01} = \mathrm{Re}\,\phi_{02} = 0$ then $|\mathcal{B}_1(t)|$ and $|\mathcal{B}_2(t)|$ may vanish and the corresponding solutions $q_1(t)$ and $q_2(t)$ become singular. Due to the periodicity, if $|\mathcal{B}_1(t)|$ and $|\mathcal{B}_2(t)|$ vanish at certain points t_{01} and t_{02}, respectively, then they will also vanish at $t_{01} + k\tau$ and $t_{02} + k\tau$ for any integer $k = 0, \pm 1, \pm 2, \ldots$.

Example 5. *Let $\mathfrak{g} \simeq \boldsymbol{C}_2 = sp(4)$. Since the algebras $\boldsymbol{B}_2 \simeq \boldsymbol{C}_2$, then the corresponding solutions differ by a change of variables. Let us denote all variables of the $\boldsymbol{C}_2$-CTC model by the same letters as for $\boldsymbol{B}_2$, adding an additional 'bar' to distinguish between them. Then the $\boldsymbol{C}_2$-CTC system has the form*

$$\frac{\mathrm{d}^2 \bar{q}_1}{\mathrm{d}t^2} = \mathrm{e}^{\bar{q}_2 - \bar{q}_1} \qquad \frac{\mathrm{d}^2 \bar{q}_2}{\mathrm{d}t^2} = -\mathrm{e}^{\bar{q}_2 - \bar{q}_1} + 2\mathrm{e}^{-2\bar{q}_2} \tag{4.34}$$

and the solution is presented by

$$\bar{q}_1(t) = \ln \bar{\mathcal{B}}_1(t) + \ln 2 \qquad \bar{q}_2(t) = \ln \frac{\bar{\mathcal{B}}_2(t)}{\bar{\mathcal{B}}_1(t)} + \tfrac{1}{2}\ln 2 \tag{4.35}$$

where

$$\bar{\mathcal{B}}_1(t) = 2\mathcal{B}_2(t, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\phi}_{01}, \bar{\phi}_{02}) \qquad \bar{\mathcal{B}}_2(t) = 2\mathcal{B}_1(t, \bar{\zeta}_1, \bar{\zeta}_2, \bar{\phi}_{01}, \bar{\phi}_{02}) \tag{4.36}$$

and $\mathcal{B}_k(t)$ *are given by (4.31) and*

$$\begin{aligned} \bar{\zeta}_1 &= \zeta_1 + \zeta_2 & \bar{\zeta}_2 &= \zeta_1 - \zeta_2 \\ \bar{\phi}_{01} &= \phi_{01} + \phi_{02} & \bar{\phi}_{02} &= \phi_{01} - \phi_{02}. \end{aligned} \tag{4.37}$$

Of course these last two examples are analytic continuations of the solutions presented in [11].

In analogy with the previous example we may assume ζ_k to be purely imaginary with $\eta_k = p_k\eta_0$ with integer p_k. Then we obtain the corresponding periodic solutions to equation (4.34). More generally, inserting purely imaginary values for ζ_k will result in a periodic solution in all the above examples. These solutions may become singular if the corresponding parameters ϕ_{0k} are purely imaginary.

4.5. Degenerate solutions

Let us briefly discuss the degenerate solutions to the CTC. The degeneracy is possible only if the matrix $L(0)$ has non-trivial Jordan cells [20].

One possibility to derive the degenerate solutions is to evaluate the limit $\zeta_1 \to \zeta_2 \to \cdots \to \zeta_k$ of the solution (3.9) and (3.10) using the l'Hospital rule.

If, in particular, we have complete degeneracy (i.e. all ζ_k are equal to zero) the solution of the $sl(N)$-CTC can be obtained in a simpler way. Since all $B_k(t)$ are polynomials in t which must satisfy

$$B_0(t) = B_N(t) = 1 \qquad \ddot{B}_k B_k - \dot{B}_k^2 = B_{k-1}B_{k+1} \tag{4.38}$$

we find that they depend on $N-1$ constants f_k, $k = 1, \ldots, N-1$. More specifically $B_k(t)$ must be a polynomial of degree $k(N-k)$ whose coefficients can be determined explicitly, for example, by the method of undefined constants.

For example, for $N = 3$ and $\zeta_1 = \zeta_2 = \zeta_3 = 0$ we obtain

$$B_1(t) = -\tfrac{1}{2}t^2 + f_1 t + f_2 \qquad B_2(t) = -\tfrac{1}{2}t^2 + f_1 t - f_1^2 - f_2 \tag{4.39}$$

and $B_3 = 1$, where $f_k, k = 1, 2$ are complex constants. If f_1 is real we may use the translational invariance of the CTC equation and change $t \to t + f_1$ to eliminate it; then the solution (4.39) can be written in the form

$$B_1(t) = -\tfrac{1}{2}t^2 + F_1 \qquad B_2(t) = -\tfrac{1}{2}t^2 - F_1 \qquad F_1 = \tfrac{1}{2}f_1^2 + f_2. \tag{4.40}$$

If F_1 is real, then the solutions $q_k(t)$ have singularities for $t = \pm\sqrt{2|F_1|}$. The large time asymptotics are given by

$$q^{\pm}_{1,\mathrm{as}}(t) = -q^{\pm}_{3,\mathrm{as}}(t) = 2\ln t - \ln 2 + \mathrm{i}\pi \qquad q^{\pm}_{2,\mathrm{as}}(t) = 0 \tag{4.41}$$

i.e. they do not depend on the constants f_k and are complex. Analogously for $N = 4$ and $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$ we find

$$\begin{aligned} &B_0(t) = B_4(t) = 1 \qquad B_1(t) = \tfrac{1}{6}(t^3 + f_1 t^2 + f_2 t + f_3) \\ &B_2(t) = -\tfrac{1}{12}t^4 - \tfrac{1}{9}f_1 t^3 - \tfrac{1}{18}f_1^2 t^2 - \tfrac{1}{18}(f_1 f_2 - 3f_3)t - \tfrac{1}{36}(f_2^2 - 2f_1 f_3) \\ &B_3(t) = -\tfrac{1}{6}t^3 - \tfrac{1}{6}f_1 t^2 - (\tfrac{1}{9}f_1^2 - \tfrac{1}{6}f_2)t - \tfrac{1}{54}(2f_1^3 - 6f_1 f_2 + 9f_3) \end{aligned} \tag{4.42}$$

where f_k, $k = 1, 2, 3$ are complex constants. If f_1 is real we may change $t \to t + f_1/3$ to eliminate one of these constants; then the solution (4.42) can be written in the form

$$\begin{aligned} B_1(t) &= \tfrac{1}{6}\big(t^3 + F_2 t + F_3\big) \\ B_2(t) &= -\tfrac{1}{36}\big(3t^4 - 6F_3 t + F_2^2\big) \\ B_3(t) &= -\tfrac{1}{6}\big(t^3 - F_2 t + F_3\big) \end{aligned} \tag{4.43}$$

where F_1 and F_2 are expressed through f_k by

$$F_2 = f_2 - \tfrac{1}{3} f_1^2 \qquad F_3 = f_3 + \tfrac{2}{27} f_1^3 - \tfrac{1}{3} f_1 f_2. \tag{4.44}$$

Obviously, these solutions will be regular if $B_k(t)$ have complex roots and will develop singularities if one (or more) of their roots are real. More specifically if $F_2 = 0$ the solution becomes symmetric, i.e. $B_1(t) = -B_3(t)$ and has a singularity at $t = 0$. If in addition F_3 is real, then there are singularities also for $t = -\sqrt[3]{F_3}$ and $t = \sqrt[3]{2F_3}$.

The asymptotics of these solutions are easy to calculate:

$$q_{1,\mathrm{as}\pm}(t) = 3\ln t - \ln 6 \qquad q_{2,\mathrm{as}\pm}(t) = \ln t - \ln 2 - \mathrm{i}\pi \tag{4.45}$$

$$q_{3,\mathrm{as}\pm}(t) = -\ln t + \ln 2 \qquad q_{4,\mathrm{as}\pm}(t) = -3\ln t + \ln 6 + \mathrm{i}\pi. \tag{4.46}$$

Note that again these asymptotics: (a) do not depend on the constants F_k and (b) are always complex. The last property is a consequence of the fact that degeneracy is only possible for the CTC.

5. Conclusions

Detailed analysis of the properties of the fundamental representations of the simple Lie algebras allowed us to propose an effective and invariant parametrization for the solutions of the CTC. These solutions describe much richer asymptotical regimes compared to the RTC. The explicit solutions proposed above allow one to evaluate explicitly the large-time asymptotics for the whole variety of dynamical regimes. The degenerate solutions also deserve further investigation.

One final remark is that one more step is necessary for the perfection of the explicit formulae (1.3), namely one should look for an invariant expression for the functions $W^{(k)}(\vec{\zeta}, \gamma_I^{(k)})$ in terms of $\gamma_I^{(k)}$ and the roots system Δ only. Work in this direction is in progress.

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Appendix A. The properties of V

Here we outline some of the details in deriving the expressions for $A_k(t)$ and $B_k(t)$. As we mentioned in section 2 we need the explicit expressions for the minors of V.

Let us consider the eigenvalue problem (2.3) and let us make use of the explicit tridiagonal form of $L(0)$. Then it is not difficult to find that the eigenvector related to ζ_k is of the form

$$v^{(k)} = \begin{pmatrix} r_k p_1 \\ r_k p_2(\zeta_k + P_0(\zeta_k)) \\ r_k p_3(\zeta_k^2 + P_1(\zeta_k)) \\ \vdots \\ r_k p_N(\zeta_k^{N-1} + P_{N-2}(\zeta_k)) \end{pmatrix} \tag{A.1}$$

where $P_s(\zeta_k)$ stands for a polynomial of degree s in ζ_k and

$$p_1 = 1 \qquad p_2 = \frac{1}{a_1(0)} \qquad p_3 = \frac{1}{a_1(0)a_2(0)} \qquad p_N = \frac{1}{a_1(0)\ldots a_{N-1}(0)}. \tag{A.2}$$

Next we note that the terms with $P_s(\zeta_k)$ do not contribute to the minors

$$V \left\{ \begin{matrix} 1 & \ldots & k \\ i_1 & \ldots & i_k \end{matrix} \right\}$$

and so

$$V \left\{ \begin{matrix} 1 & \ldots & k \\ i_1 & \ldots & i_k \end{matrix} \right\} = r_{i_1} \ldots r_{i_k} p_1 \ldots p_k W(i_1, \ldots, i_k) \tag{A.3}$$

where

$$W(i_1, \ldots, i_k) = \det \begin{vmatrix} 1 & \ldots & 1 \\ 2\zeta_{i_1} & \ldots & 2\zeta_{i_k} \\ \vdots & & \vdots \\ (2\zeta_{i_1})^{k-1} & \ldots & (2\zeta_{i_k})^{k-1} \end{vmatrix} = \prod_{s>p; s,p \in I} 2(\zeta_s - \zeta_p) \tag{A.4}$$

is the Vandermonde determinant and $I = \{i_1, \ldots, i_k\}$. Next we have to take care of the factors p_s, which can be expressed through $\vec{q}(0)$ since $a_k(0) = \frac{1}{2}\exp(-(\vec{q}(0), \alpha_k)/2)$. We also note that r_k are determined up to a sign by the normalization condition (2.4). These remarks and the properties of the fundamental representations of the series $\boldsymbol{A}_r$ (2.19) and (2.20) are sufficient to treat the $\boldsymbol{A}_r$ series.

Let us now derive the symmetry relations (2.9) and (2.10) for the $\boldsymbol{B}_r$ and $\boldsymbol{C}_r$ algebras. To this end we introduce the S matrices as follows:

$$\begin{aligned} S &= \sum_{k=1}^{r} (-1)^{k+1} \left(E_{k\bar{k}} + E_{\bar{k}k}\right) + (-1)^r E_{r+1,r+1} && \text{for} \quad \boldsymbol{B}_r \\ &= \sum_{k=1}^{r} (-1)^{k+1} \left(E_{k\bar{k}} - E_{\bar{k}k}\right) && \text{for} \quad \boldsymbol{C}_r \\ &= \sum_{k=1}^{r} (-1)^{k+1} \left(E_{k\bar{k}} + E_{\bar{k}k}\right) && \text{for} \quad \boldsymbol{D}_r \end{aligned} \tag{A.5}$$

which enter into the definition of the corresponding orthogonal and symplectic algebras. By E_{jk} we denote an $N \times N$ matrix whose matrix elements are equal to $(E_{jk})_{mn} = \delta_{jm}\delta_{kn}$ and as in (2.9) $\bar{k} = N + 1 - k$. Then we make use of the fact that if V is a group element of the corresponding group then $V^T = SV^{-1}S^{-1}$, i.e.

$$r_k = V \left\{ \begin{matrix} 1 & \ldots & \hat{\bar{k}} & \ldots & N \\ 1 & \ldots & k & \ldots & N-1 \end{matrix} \right\} \tag{A.6}$$

where the 'hat' means that the index $\bar{k}$ is missing. Equations (A.3) and (A.6) readily give

$$r_k r_{\bar{k}} = \frac{W(1, \ldots, \hat{\bar{k}}, \ldots, N)}{p_{\bar{k}} W(1, \ldots, N)} \prod_{s=1}^{N} p_s r_s. \tag{A.7}$$

Taking the product of (A.7) for $k = 1$ to r one is able to evaluate $\prod_{s=1}^{N} r_k$ in terms of p_s and ζ_s alone. Next putting p_s, N and ζ_s as appropriate for the series $\boldsymbol{B}_r$ and $\boldsymbol{C}_r$ we derive the expressions for w_k (2.11)–(2.13).

For the $\boldsymbol{D}_r$ series the matrix $L(0)$ is of the form

$$L(0) = \left(\begin{array}{ccccc|ccccc}
b_1 & a_1 & & & & & & & & \\
a_1 & b_2 & & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & \\
 & & & b_{r-1} & a_{r-1} & a_r & 0 & & & \\
 & & & a_{r-1} & b_r & 0 & a_r & & & \\
\hline
 & & & a_r & 0 & -b_r & a_{r-1} & & & \\
 & & & 0 & a_r & a_{r-1} & -b_{r-1} & & & \\
 & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & & -b_2 & a_1 \\
 & & & & & & & & a_1 & -b_1
\end{array}\right). \tag{A.8}$$

For the kth eigenvector we obtain

$$v^{(k)} \simeq \begin{pmatrix} r_k p_1 \\ r_k p_2 \zeta_k \\ \vdots \\ r_k p_{r-1} \zeta_k^{r-2} \\ r_k p_r (\zeta_k^{r-1} + C/\zeta_k) \\ r_k p_r (\zeta_k^{r-1} - C/\zeta_k) \\ r_k \zeta_k^r p_{r+2} \\ \vdots \\ r_k \zeta_k^{2r-2} p_{2r} \end{pmatrix} \tag{A.9}$$

where C is a coefficient to be calculated below. The symbol $\simeq$ in (A.9) means that in the right-hand side we have omitted terms polynomial in ζ_k which do not contribute to the minors of V. Note also that C enters into play only when we need a minor of order r or higher. Such a necessity appears in two cases: when we evaluate the expression (3.32) for $\tilde{A}_r(t)$ and when we derive the symmetry relation.

From (2.3) and the explicit formula for $L(0)$ (A.8) we find that p_k in (A.9) are given as follows:

$$\begin{gathered}
p_1 = 1 \qquad p_k = \prod_{s=1}^{k-1} \frac{1}{a_s(0)} \qquad \text{for} \quad k = 2, \ldots, r-1 \\
p_r = \frac{1}{2} \prod_{s=1}^{r-1} \frac{1}{a_s(0)} \qquad p_{r+1} = \frac{1}{2a_r(0)} \prod_{s=1}^{r-2} \frac{1}{a_s(0)} \qquad p_{r+2} = \frac{1}{2} \prod_{s=1}^{r} \frac{1}{a_s(0)} \\
p_{r+k} = p_{r+2} \prod_{s=2}^{k-1} \frac{1}{a_{r-s}(0)} \qquad \text{for} \quad k = 3, \ldots, r.
\end{gathered} \tag{A.10}$$

The determinant of V gives

$$1 = \det V = 2(-1)^{r-1} C\, W(1, \ldots, 2r) \prod_{s=1}^{2r} \frac{r_s p_s}{\zeta_s}. \tag{A.11}$$

In analogy with (A.6) and (A.7) we obtain

$$r_p r_{\bar{p}} = \frac{(-1)^r 2C\zeta_p}{p_{2r}} \frac{W(1, \ldots, \hat{\bar{p}}, \ldots, 2r)}{W(1, \ldots, 2r)} \prod_{s=1}^{2r} \frac{p_s r_s}{\zeta_s}. \tag{A.12}$$

Taking again the product $\prod_{p=1}^{r} r_p r_{\bar{p}}$ in (A.12) and substituting the expressions for p_s from (A.10) we find

$$C = (-1)^{r+1} \prod_{s=1}^{r} \zeta_s \tag{A.13}$$

and, in addition, the relation (2.14).

Now it is easy to find the expression for the minor of order r:

$$V \left\{ \begin{array}{ccc} 1 & \ldots & r \\ i_1 & \ldots & i_r \end{array} \right\} = \frac{1}{2} \left(1 + \frac{\zeta_1 \ldots \zeta_r}{\zeta_{i_1} \ldots \zeta_{i_r}} \right) W(i_1, \ldots, i_r) \prod_{s=1}^{r} r_{i_s} p_s \tag{A.14}$$

needed for the derivation of $\tilde{A}_r(t)$ (3.32).

Appendix B. Algebraic details

The action of w_0 on the simple roots is well known [24, 25]:

$$w_0(\alpha_k) = -\alpha_{\tilde{k}} \tag{B.1}$$

where $\tilde{k} = r - k + 1$ for $\boldsymbol{A}_r$; $\tilde{k} = k$, $k = 1, \ldots, r$ for $\boldsymbol{B}_r, \boldsymbol{C}_r$ and $\boldsymbol{D}_{2n}$. For $\mathfrak{g} \simeq \boldsymbol{D}_{2n+1}$ we have $\tilde{k} = k$ for $k \leqslant 2n - 1$ and $w_0(\alpha_{2n}) = -\alpha_{2n+1}$, $w_0(\alpha_{2n+1}) = -\alpha_{2n}$. More specifically w_0 acts on the orthonormal basis $\{e_k\}$ in the root space as follows:

$$\begin{aligned} w_0(e_k) &= e_{\tilde{k}} && \text{for} \quad \boldsymbol{A}_r \\ w_0(e_k) &= -e_k && \text{for} \quad \boldsymbol{B}_r, \boldsymbol{C}_r, \boldsymbol{D}_{2n} \end{aligned} \tag{B.2}$$

and for $\boldsymbol{D}_{2n+1}$

$$w_0(e_k) = -e_k \qquad \text{for} \quad k = 1, \ldots, 2n \qquad w_0(e_{2n+1}) = e_{2n+1}.$$

Next, it is well known that the weight system $\Gamma(\omega)$ is determined uniquely by the highest weight ω. The reconstruction of the weights $\gamma \in \Gamma(\omega)$ is performed by using two facts:

(i) if $\gamma \in \Gamma(\omega)$ then $w(\gamma) \in \Gamma(\omega)$ where w is any element of the Weyl group; besides γ and $w(\gamma)$ have equal multiplicities;

(ii) if $\alpha > 0$ is a positive root and

$$\frac{2(\alpha, \omega)}{(\alpha, \alpha)} = p > 0 \tag{B.3}$$

then $\omega - s\alpha \in \Gamma(\omega)$ for all $s = 1, \ldots, p$.

In particular, if $\omega = \omega_k^+$ and $\alpha = \sum_{s=1}^r m_s\alpha_s$ we see that (B.3) is fulfilled only if $m_k = p > 0$. Thus we find that generically (i.e. for $k < r$) along with ω_k^+, weights in $\Gamma(\omega_k^+)$ are also

$$\gamma_1 = \omega_k^+ - \alpha_k \qquad \gamma_2 = \omega_k^+ - (\alpha_{k-1} + \alpha_k) \qquad \text{etc.}$$

Using the sorting condition (4.2) we easily find that

$$\min_{\gamma\in\Gamma(\omega_k^+)\backslash\omega_k^+} \left[-(\vec{\kappa}, \omega_k^+ - \gamma)\right] = \min\left[\sum_{s=1}^r m_s(-\vec{\kappa}, \alpha_s)\right] = -(\vec{\kappa}, \alpha_k) \tag{B.4}$$

which proves the estimations in equations (4.3).

The same method can also be applied when one of the roots satisfies $(\vec{\kappa}, \alpha_m) = 0$. As a consequence of this condition at least two terms in $\mathcal{B}_k(t)$ may have the same asymptotic behaviour.

Here we will first describe the sets of roots $G_p^+(\vec{\kappa})$ (see (4.14)) and then will also outline the proof of (4.16). Obviously if $(\vec{\kappa}, \alpha_m) = 0$ only $G_m^+(\vec{\kappa})$ will be non-empty and will coincide with $\{\alpha_m\}$ and therefore $\Gamma_{p,+}(\omega_p^+)\backslash\{\omega_p^+\}$, while $\Gamma_{m,+}(\omega_m^+)\backslash\{\omega_m^+, \omega_m^+ - \alpha\}$ where $\alpha = \alpha_m + \sum_s m_s\alpha_s$. The minimum of $-2(\vec{\kappa}, \omega_m^+ - \alpha)$ will be achieved if we limit ourselves with roots α of height 2. Now it remains to take into account that $\alpha_m + \alpha_s$ is a root if and only if $(\alpha_m, \alpha_s) < 0$. The corresponding result for $t \to -\infty$ is obtained by acting with w_0. This proves (4.16).

We finish this appendix by describing the sets of roots $G_p^+(\vec{\kappa})$ for $(\vec{\kappa}, \alpha_m) = (\vec{\kappa}, \alpha_m) = 0$. First, if $(\alpha_m, \alpha_p) = 0$ then $G_m^+(\vec{\kappa}) = \{\alpha_m\}$, $G_p^+(\vec{\kappa}) = \{\alpha_p\}$ and all the others $G_k^+(\vec{\kappa}) = \{\emptyset\}$. If, however, $(\alpha_m, \alpha_p) < 0$ the situation becomes more interesting. In the generic case $(\alpha_m, \alpha_p) = -1$ we find

$$G_m^+(\vec{\kappa}) = \{\alpha_m, \alpha_m + \alpha_p\} \qquad G_p^+(\vec{\kappa}) = \{\alpha_p, \alpha_m + \alpha_p\}.$$

The only two exceptions of this rule for the classical series are $m = r - 1$, $p = r$ for $\mathfrak{g} \simeq \boldsymbol{B}_r$ and $\boldsymbol{C}_r$. Then we have

$$G_{r-1}^+(\vec{\kappa}) = \{\alpha_{r-1}, \alpha_{r-1} + \alpha_r, \alpha_{r-1} + 2\alpha_r\} \qquad G_r^+(\vec{\kappa}) = \{\alpha_r, \alpha_{r-1} + \alpha_r, \alpha_{r-1} + 2\alpha_r\}$$

for $\mathfrak{g} \simeq \boldsymbol{B}_r$ and

$$G_{r-1}^+(\vec{\kappa}) = \{\alpha_{r-1}, \alpha_{r-1} + \alpha_r, 2\alpha_{r-1} + \alpha_r\} \qquad G_r^+(\vec{\kappa}) = \{\alpha_r, \alpha_{r-1} + \alpha_r, 2\alpha_{r-1} + \alpha_r\}$$

for $\mathfrak{g} \simeq \boldsymbol{C}_r$. These last relations allow us to calculate the asymptotics of $\mathcal{B}_{k,\text{as}}^{\pm}$ for all possible values of k for $I_{\text{bs}} = \{m, p\}$. Then it is not difficult to insert them in (4.6) and evaluate the asymptotic behaviour of all $q_k(t)$. Several examples of such calculations were presented in section 4 above.

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